# MTH6104 / MTH6104P: Algebraic structures II

**Duration: 2 hours**

**Date and time: 28th May 2015, 2:30pm**

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**You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.**

**Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.**

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**Examiner(s): Matthew Fayers**

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In this paper, we use the following notation.

- $C_n$ denotes the cyclic group of order $n$.
- $\mathcal{U}_n$ is the set of integers between 0 and $n$ which are prime to $n$, with the group operation being multiplication modulo $n$.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

  $$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$ (with the group operation being composition).
- If $p$ is a prime, then $\mathbb{F}_p$ is the set $\{0, 1, \ldots, p-1\}$, with addition and multiplication modulo $p$. $\mathrm{GL}_2(\mathbb{F}_p)$ is the group of $2\times2$ invertible matrices with entries in $\mathbb{F}_p$, with the group operation being matrix multiplication.

**Question 1.**

(a) Give the definition of a **group**. **[3]**

Suppose $G$ is a group and $g \in G$. In the rest of this question you may use elementary rules for manipulating powers of elements.

(b) Give the definition of the **cyclic subgroup** $\langle g \rangle$, and prove that it is a subgroup of $G$. **[6]**

(c) In the case where $G = \mathcal{U}_{25}$, find $\langle 6 \rangle$. **[4]**

(d) Give the definition of the **order** of $g$. **[2]**

(e) Suppose $\mathrm{ord}(g) = n < \infty$, and $m$ is an integer such that $g^m = 1$. Prove that $n$ divides $m$. **[5]**

(f) Suppose $\mathrm{ord}(f) = 3$, $\mathrm{ord}(g) = 2$ and $fg = gf$. What is the order of $fg$? Justify your answer. **[5]**

**Question 2.** Write an essay on group homomorphisms. *[You should include precise definitions and statements of results, illustrated by examples, and give some proofs.]* **[25]**

**Question 3.**

(a) Give the definition of a **normal** subgroup and the definition of a **simple** group. *[You do not need to define what a group or a subgroup is.]* **[4]**

(b) Suppose $G$ is an abelian group of order 60. Prove that $G$ is not simple. *[You may use basic results about the orders of elements of G.]* **[6]**

(c) Now suppose $G$ is a finite group and $p$ is a prime. Give the definition of a **Sylow $p$-subgroup** of $G$. **[2]**

(d) Write down a Sylow 2-subgroup, a Sylow 3-subgroup and a Sylow 5-subgroup of $\mathcal{U}_{27}$. **[6]**

(e) State Sylow's Theorem 3 concerning the number of Sylow $p$-subgroups of a finite group. **[3]**

(f) Using this theorem, prove that there is no simple group of order 63. **[4]**

**Question 4.** Suppose $G$ is a group, $H$ is a subgroup of $G$ and $f, g \in G$.

(a) Define the **right coset** $Hg$, and the **index** of $H$ in $G$. **[4]**

(b) Give a precise statement of Lagrange's Theorem. **[2]**

(c) Find all the right cosets of $\{1, r^3, rs, r^4s\}$ in $\mathcal{D}_{12}$. **[5]**

(d) Suppose $H$ is a normal subgroup of $G$. Prove that $Hg = gH$ for every $g \in G$. **[5]**

(e) Hence show that $\langle(1\ 2\ 3\ 4)\rangle$ is not a normal subgroup of $\mathcal{S}_4$. **[3]**

(f) Now let

$$X = \left\{ g \in \mathcal{S}_{10} \;\middle|\; g \cdot 1 = 2 \right\}.$$

Find a subgroup $H$ of $\mathcal{S}_{10}$ and $g \in \mathcal{S}_{10}$ such that $X = Hg$. Justify your answer. *[In particular, you should prove that H is a subgroup of $\mathcal{S}_{10}$.]* **[6]**

 

**Question 5.** Suppose $G$ is a group.

(a) Define what it means for two elements of $G$ to be **conjugate** in $G$. **[2]**

(b) Show that $(1\ 2\ 3)(4\ 5)$ and $(1\ 2\ 5)(3\ 4)$ are conjugate in $\mathcal{S}_5$. **[3]**

(c) Give the definition of the **centre** of $G$. **[2]**

(d) Find (with proof) the centre of $\mathcal{D}_{10}$. **[7]**

(e) Suppose $G/Z(G)$ is a cyclic group. Prove that $G$ is abelian. **[5]**

(f) Suppose $g \in G$. Give the definition of the **centraliser** $C_G(g)$. **[2]**

(g) Suppose $g \in G$ but $g \notin Z(G)$. Show that $Z(G) \neq C_G(g) \neq G$. **[4]**

**Question 6.** Suppose $G$ is a group and $X$ is a set.

(a) Give the definition of an **action** of $G$ on $X$. **[3]**

Suppose $\pi$ is an action of $G$ on $X$, and define a relation $\equiv$ on $X$ by setting $x \equiv y$ if there is $g \in G$ such that $\pi_g(x) = y$.

(b) Prove that $\equiv$ is an equivalence relation on $X$. **[4]**

(c) Suppose $x \in X$. Give the definitions of the **orbit** and the **stabiliser** of $x$ under $\pi$. **[4]**

(d) Give an example of a transitive action of $C_4$ on $C_4$. **[3]**

(e) Give a precise statement of the Orbit–Stabiliser Theorem. **[3]**

Now let $G = \mathrm{GL}_2(\mathbb{F}_{11})$. Let $X$ be the set of non-zero column vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{F}_{11}$. Define an action of $G$ on $X$ by $\pi_g(x) = gx$. (You may assume that $\pi$ really is an action.)

(f) Prove that $\pi$ is transitive. **[4]**

(g) Hence use the Orbit–Stabiliser Theorem to find $|G|$. **[4]**

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**End of Paper.**