Queen Mary
University of London

MTH6104: Algebraic structures II

Duration: 2 hours

Date and time: 16th May 2014, 10:00 a.m.

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): Matthew Fayers

In this paper, we use the following notation.

- C_n denotes the cyclic group of order n.
- Q_8 is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which
$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$
- $\mathrm{GL}_2(\mathbb{C})$ is the group of invertible 2×2 matrices over $\mathbb{C}$, with the group operation being matrix multiplication.

Question 1.

(a) Give the definition of a **group**, and a **subgroup**. **[5]**

Suppose G is a group and $g \in G$.

(b) Give the definition of the **subgroup of** G **generated by** g, and prove that it is a subgroup of G. *[You may use elementary rules for manipulating powers of elements.]* **[6]**

(c) Give the definition of the **order** of g. **[2]**

For the rest of this question, let G be the following group of order 21:

$$G = \left\{1, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2\right\},$$

where a is an element of order 7, b is an element of order 3 and $ba = a^2b$.

(d) Find three elements of G which are conjugate to ab. **[5]**

(e) Find the elements of $\langle ab \rangle$. **[4]**

[For parts (d) and (e) you should write elements of G in the form a^ib^j as in the above list.]

(f) Is $\langle ab \rangle$ a normal subgroup of G? Justify your answer. **[3]**

Question 2.

(a) Give the definition of the **symmetric group** $\mathcal{S}_n$. **[3]**

(b) Suppose $f, g \in \mathcal{S}_5$ are defined by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}.$$

Write down g, g^{-1} and fgf^{-1} in disjoint cycle notation. **[4]**

(c) Give the definition of a **transposition**. **[2]**

(d) Give the definition of the **alternating group** $\mathcal{A}_n$, and prove that $\mathcal{A}_n \trianglelefteq \mathcal{S}_n$. **[7]**

(e) Suppose $h \in \mathcal{S}_n$. Explain how you can use the disjoint cycle notation for h to find the order of h and to find whether $h \in \mathcal{A}_n$. **[3]**

(f) Suppose $N \trianglelefteq \mathcal{A}_6$, and that N contains the element $(1\ 3\ 5)(2\ 4\ 6)$. Prove that N contains a 3-cycle. *[You may not assume that $\mathcal{A}_6$ is simple.]* **[6]**

Question 3. Suppose G is a group.

(a) Suppose X is a set. Give the definition of an **action** of G on X. **[3]**

(b) Suppose n is a positive integer, and let X be the set of all subsets of G of size n. Define an action of G on X by

$$\pi_g(\{g_1, \dots, g_n\}) = \{gg_1, \dots, gg_n\}.$$

Prove that π really is an action. **[5]**

(c) Suppose π is an action of G on X, and $x \in X$. Give the definition of the **orbit** of x and the **stabiliser** of x. **[4]**

(d) State and prove the Orbit–Stabiliser Theorem. *[You may assume that a stabiliser is a subgroup, and you may use Lagrange's Theorem.]* **[8]**

(e) Use the Orbit–Stabiliser Theorem to find the order of the symmetry group of a cube. **[5]**

Question 4. *[In this question, you may assume any results you need about actions of groups.]* Suppose p is a prime.

(a) What does it mean to say that a finite group is a p-**group**? **[2]**

(b) Describe a group G of order p^2 which is not isomorphic to C_{p^2}. *[You should be explicit about what the underlying set and the binary operation are, but you do not have to prove that G is a group.]* Explain how you know that G is not isomorphic to C_{p^2}. **[4]**

(c) Give the definition of a **Sylow** p-**subgroup**. **[2]**

(d) Give precise statements of all the Sylow Theorems. **[7]**

(e) Using one of the Sylow Theorems, show that:

- there is only one group of order 87 up to isomorphism, and
- there is no simple group of order 56.

[You may assume a general result relating the order of a group to the orders of its elements.] **[10]**

Question 5. Suppose G is a group, $g \in G$ and $N, H \leqslant G$.

(a) Given the definition of the **left coset** gH. **[2]**

(b) Prove that if $a, b, c \in gH$ then $ab^{-1}c \in gH$. **[3]**

(c) Give the definition of the set NH. **[2]**

(d) Prove that if $N, H \trianglelefteq G$, then $NH \trianglelefteq G$. **[7]**

(e) Give an example of a group G with $N, H \leqslant G$ such that NH is not a subgroup of G. *[You do not have to prove that N and H are subgroups, but you should prove that NH is not a subgroup.]* **[5]**

(f) Now suppose $G = \mathrm{GL}_2(\mathbb{C})$, and let

$$N = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle, \qquad H = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle.$$

Find all the left cosets of H in NH (which is a subgroup of G in this case). **[6]**

Question 6. Suppose G and H are groups.

(a) Give the definitions of the following:

- a **homomorphism** from G to H;
- an **isomorphism** from G to H;
- an **automorphism** of G;
- the **automorphism group** of G. **[5]**

(b) Suppose $\phi : G \to H$ is a homomorphism, and $L \leqslant H$. Prove that $\phi(\phi^{-1}(L)) = L \cap \operatorname{Im}\phi$. **[4]**

(c) Give a precise statement of the Correspondence Theorem. **[4]**

(d) Let $G = C_{50}$. Find all the subgroups of G, and draw a diagram showing which subgroups contain which others. *[You do not have to prove anything.]* **[4]**

(e) Let $\phi : C_{50} \to C_{50}$ be the homomorphism which sends g to g^5 for every $g \in C_{50}$. Find $\operatorname{Im}\phi$ and $\ker(\phi)$, and show how subgroups correspond under the Correspondence Theorem. *[You do not have to prove anything.]* **[4]**

(f) Give an example of an outer automorphism ϕ of Q_8 such that $\phi(i) = i$. *[You do not have to prove anything, but should you say where each element of Q_8 maps to.]* **[4]**

End of Paper.