MTH744u/P (Jan 2023)

Q1 (a)(i) $\dot{x} = \sin(x) - \sinh(x) \ (= f(x))$

$f(0) = 0$ and $f'(x) = \cos x - \cosh(x) \leq \cos(x) - 1 \leq 0.$

$\therefore$ f is decreasing and $f(x) > 0,\ x < 0$ & $f(x) < 0,\ x > 0$

$\therefore$ $x = 0$ is globally stable

seen examples like this

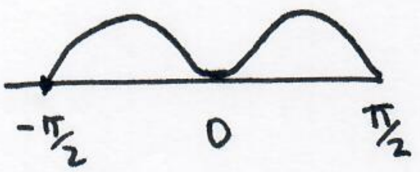
(ii) $\dot{x} = \cos x(1 - \cos x) (= f(x))$

$\operatorname{Sgn}(f(x)) = \operatorname{sgn}(\cos x)$ except for $x = n\pi$. (double zero)

So $\dot{x} > 0$ on $x \in (-\pi/2, \pi/2) \setminus \{0\}$, + translates by 2π

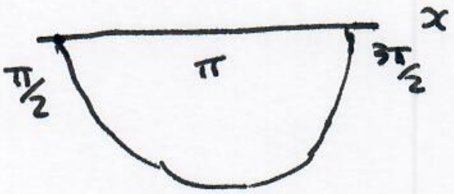
$\dot{x} < 0$ on $x \in (\pi/2, 3\pi/2)$ + translates by 2π.

Seen exs like this

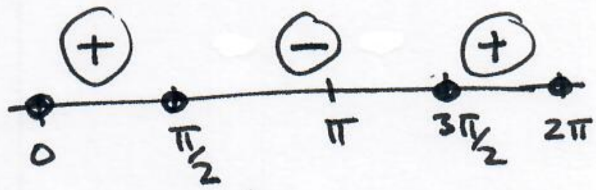


+ translates of 2π

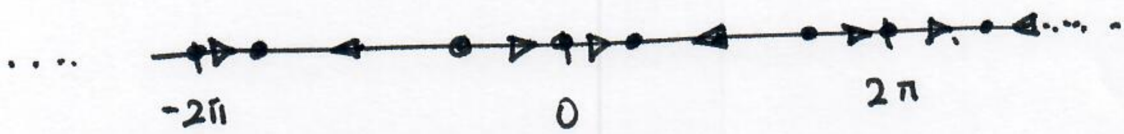
and



We obtain a sequence



We get the following phase portrait



(b) The diagram indicates a possible scenario:

$\dot{x} > 0 \qquad x < 0$

$\dot{x} > 0 \qquad 0 < x < 1$

$\dot{x} > 0 \qquad 1 < x < 2$

$\dot{x} > 0 \qquad 2 < x$

seen examples like this

Possible system

$$\dot{x} = (x-0)(x-1)(x-2)^2$$

(c) $$(x-0)(x-1)^2(x-2)^2(x-3)(x-4)^2$$

3

(d) Let $x_0 = \sup\{x \mid f(x) = 0\}$. Either $x_0 = \max\{x \mid f(x) = 0\}$ or $\exists\, x_n$, $f(x_n) = 0$ such that $x_n \to x_0$ as $n \to \infty$. f is differentiable and $\therefore$ continuous

$\Rightarrow f(x_0) = 0$

Moreover, $f(x) \neq 0$ for $x > x_0$. Note $f(x) > 0$ for $x > X_0$. $\therefore$ $f(x) > 0$ for $x > x_0$, otherwise contradiction by intermediate value theorem.

Not seen

(e) for $\dot{\theta} = f(\theta)$,

$\dot{\theta} > 0 \quad 0 \leq \theta < \pi/2$

$\dot{\theta} > 0 \quad \pi/2 < \theta < \pi$

$\dot{\theta} < 0 \quad \pi < \theta < 3\pi/2$

$\dot{\theta} < 0 \quad 3\pi/2 < \theta < 2\pi$

So $\dot{\theta} = \sin\theta$ gives $\dot{\theta} > 0 \quad 0 < \theta < \pi$

and $\dot{\theta} < 0 \quad \pi < \theta < 2\pi$

but has only two fixed points at $\theta = 0$ & $\theta = \pi$ on S^1. Introduce multiplicative factors which are positive except for $\theta = \pi/2$ & $\theta = 3\pi/2$.

Consider $\dot{\theta} = \sin\theta\left(1 - \cos\left(\theta - \pi/2\right)\right)\left(1 - \cos\left(\theta - 3\pi/2\right)\right)$

unseen example
but technique for
"squeezing" function
shown in tutorial

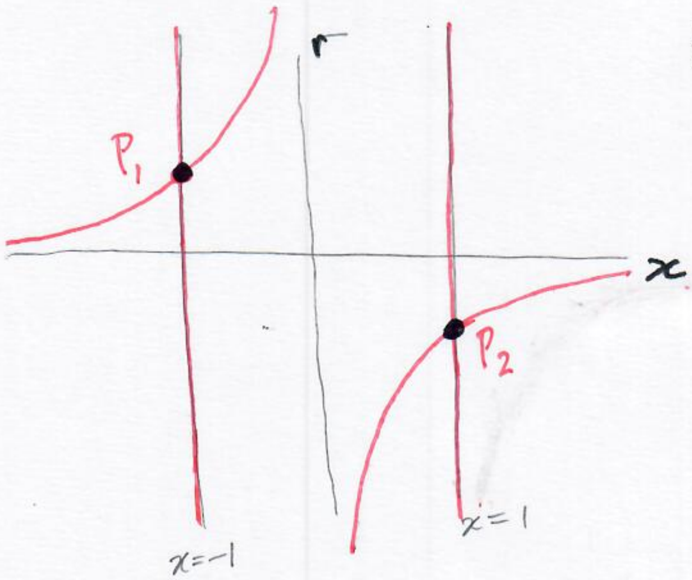
Q2

(a) (i) $\dot{x} = 1 + rx - x^2 - rx^3 \quad \left(= f_r(x)\right) \qquad (1)$

$$\dot{x} = (1+rx)(1-x^2)$$

Fixed pt sets are given by either

$$1 + rx = 0 \quad \text{or} \quad 1 - x^2 = 0$$



Fixed pts in xr-plane

Examples seen of (a) (i) – (iv)
a(v) only briefly mentioned "singular" perturbations

(ii)

$P_1 = (-1, 1)$ & $P_2(1, -1)$ have double roots of $f(x)$.

(iii) $\Rightarrow$ P_1, P_2 are potential bifurcation pts for (1) above.

Taylor expansion of (1)

Let $y = x + 1$, $\mu = r - 1$, then

$$f_r(x) = f(x,r) = 1 + (\mu+1)(y-1) - (y-1)^2 - (\mu+1)(y-1)^3$$

$$= 1 + \mu y - \mu + y - 1 - y^2 + 2y - 1$$
$$- (\mu+1)(y^3 - 3y^2 + 3y - 1)$$

$$= \boxed{1} + \mu y - \mu + y \boxed{-1} - y^2 + 2y \boxed{-1}$$
$$- \mu y^3 + 3\mu y^2 - 3\mu y + \mu$$
$$- y^3 + 3y^2 - 3y \boxed{+1}$$

Collecting terms $= \mu y - y^2 - \mu y^3 + 3\mu y^2 - 3\mu y$
$- y^3 + 3y^2$

$= -2\mu y + 2y^2 - \mu y^3 + 3\mu y^2 - y^3$

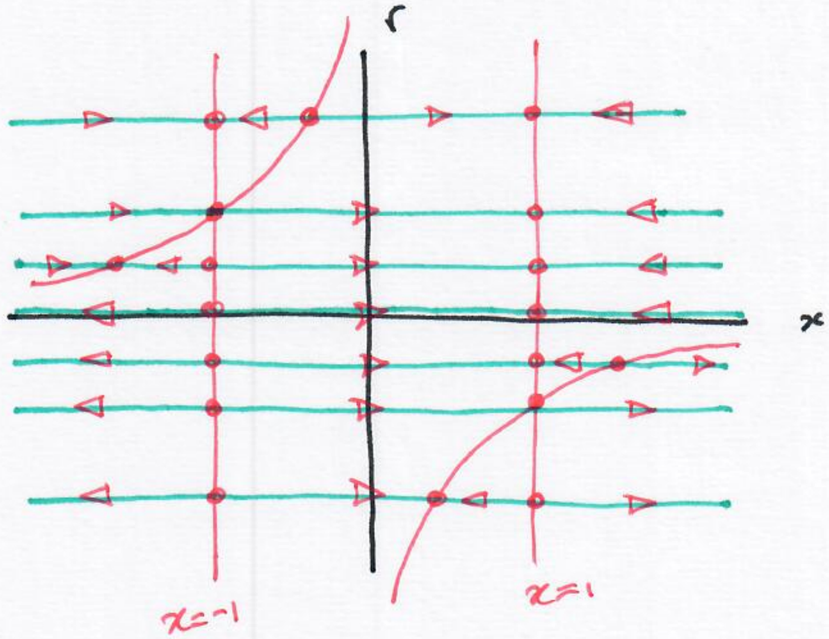
So we get $A = \frac{\partial f}{\partial r}(P_1) = 0$, $B \frac{\partial^2 f}{\partial x^2}(P_1) = 2$

and $C = \frac{\partial^2 f}{\partial x \partial r}(P_1) = -2$

$\therefore$ Transcritical at P_1

Similar analysis at P_2 also gives transcritical with $A = 0$, $B = 2$, $C = -2$.

(iv)



(v) As $r \to 0$ from either < 0 or > 0, one of the fixed points moves off to $\infty / -\infty$ (for $r<0$ / $r>0$)

(b) $\dot{x} = A\cancel{x}r + Bx^2 + Cxr = f(x,r)$, $A, B \neq 0$ 6

Note $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 2Bx + Cr \Big|_{(0,0)} = 0.$

So we have a potential bifurcation.

Let $y = \left(x + \frac{Cr}{2B}\right)$, then

$$\dot{y} = \dot{x} = Ar + B\left(y - \frac{Cr}{2B}\right)^2 + C\left(y - \frac{Cr}{2B}\right)r$$

$$= Ar + B\left(y^2 - \cancel{\frac{Cry}{B}} + \frac{C^2r^2}{4B^2}\right) + \cancel{Cyr} - \frac{C^2r^2}{2B}$$

$$= Ar + By^2 + \frac{C^2r^2}{4B} - \frac{C^2r^2}{2B}$$

$$= Ar - \frac{C^2r^2}{2B} + By^2$$

Let $y = \alpha z$

$$\dot{y} = \alpha\dot{z} = \alpha\left(Ar - \frac{C^2r^2}{2B}\right) + B\alpha^2 z^2$$

Let $B\alpha = 1$, i.e $\alpha = 1/B$, ($B \neq 0$ given).

$$\dot{z} = \frac{A}{B}r - \frac{C^2r^2}{2B^2} + z^2$$

Let $\nu = \frac{A}{B}r - \frac{C^2r^2}{2B^2}$ (non-singular change of parameter at $r=0$

$$\frac{\partial \nu}{\partial r}\Big|_{r=0} = \frac{A}{B} \neq 0.$$

$$\therefore \quad \dot{z} = \nu + z^2.$$

Completing the square mentioned but not seen in detail

Q3

(a) $\dot{x} = y, \quad \dot{y} = -x^3 + x + y$

Fixed pts $\dot{x} = 0, \dot{y} = 0$

$$y = 0, \quad x^3 = x \quad \Rightarrow \quad x = 0, \pm 1$$

Jacobian matrix $J = \begin{bmatrix} 0 & 1 \\ -3x^2+1 & 1 \end{bmatrix}$

at $(0,0)$: $J = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ E-eq^n $\lambda^2 - \lambda - 1 = 0$

Eigenvalues $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$, $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$

Eigenvectors $v_1 = (\frac{1}{2}(-1+\sqrt{5}), 1) = (0.62, 1)$ & $v_2 = (\frac{1}{2}(-1-\sqrt{5}), 1) = (-1.62, 1)$

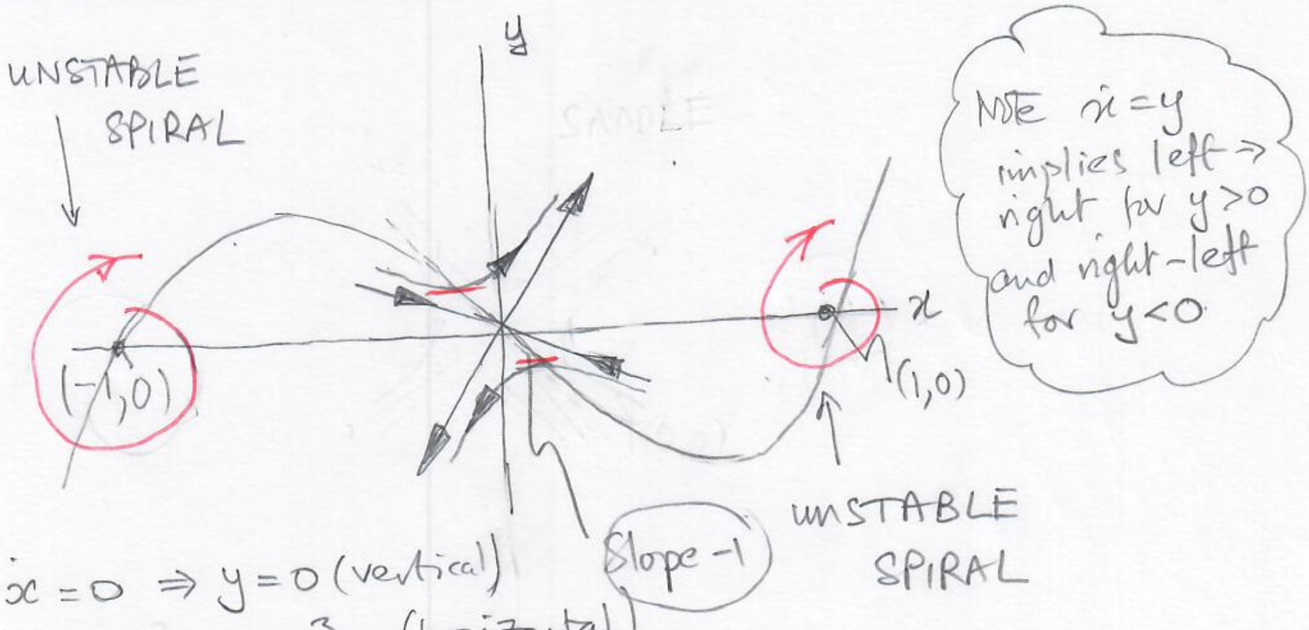
at $(1,0)$: $J = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ E$\cdot$eq^n $\lambda^2 - \lambda + 2 = 0$

Eigenvalues: $\lambda_1 = \frac{1}{2}(1+i\sqrt{7})$ $\quad \lambda_2 = \frac{1}{2}(1-i\sqrt{7})$

Eigenvectors: $v_1 = \frac{1}{4}(1 - i\sqrt{7})$, $v_2 = \frac{1}{4}(1+i\sqrt{7})$

at $(-1,0)$: same as $(1,0)$

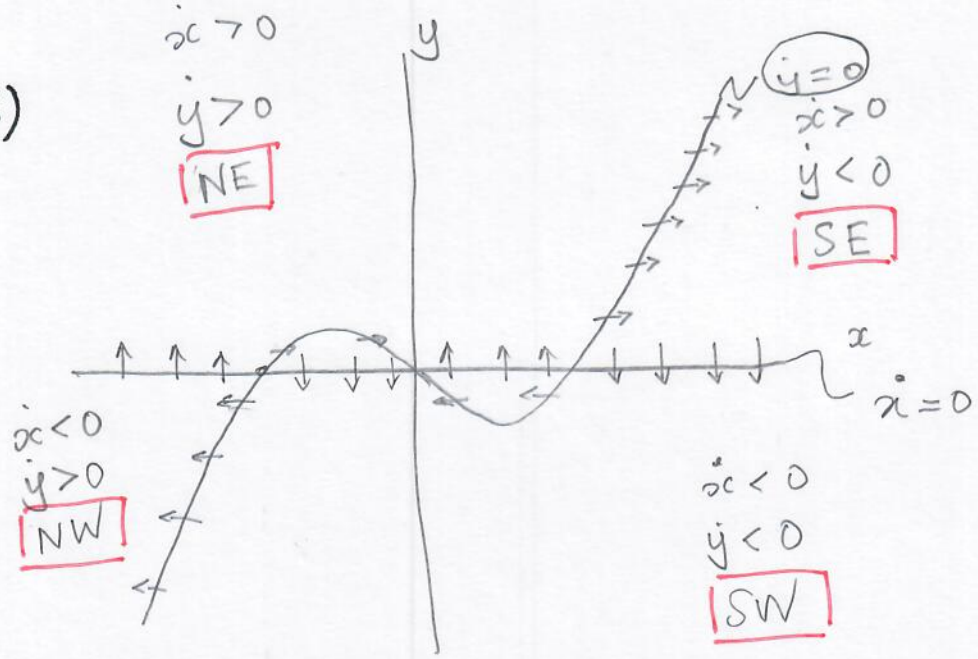
(b)



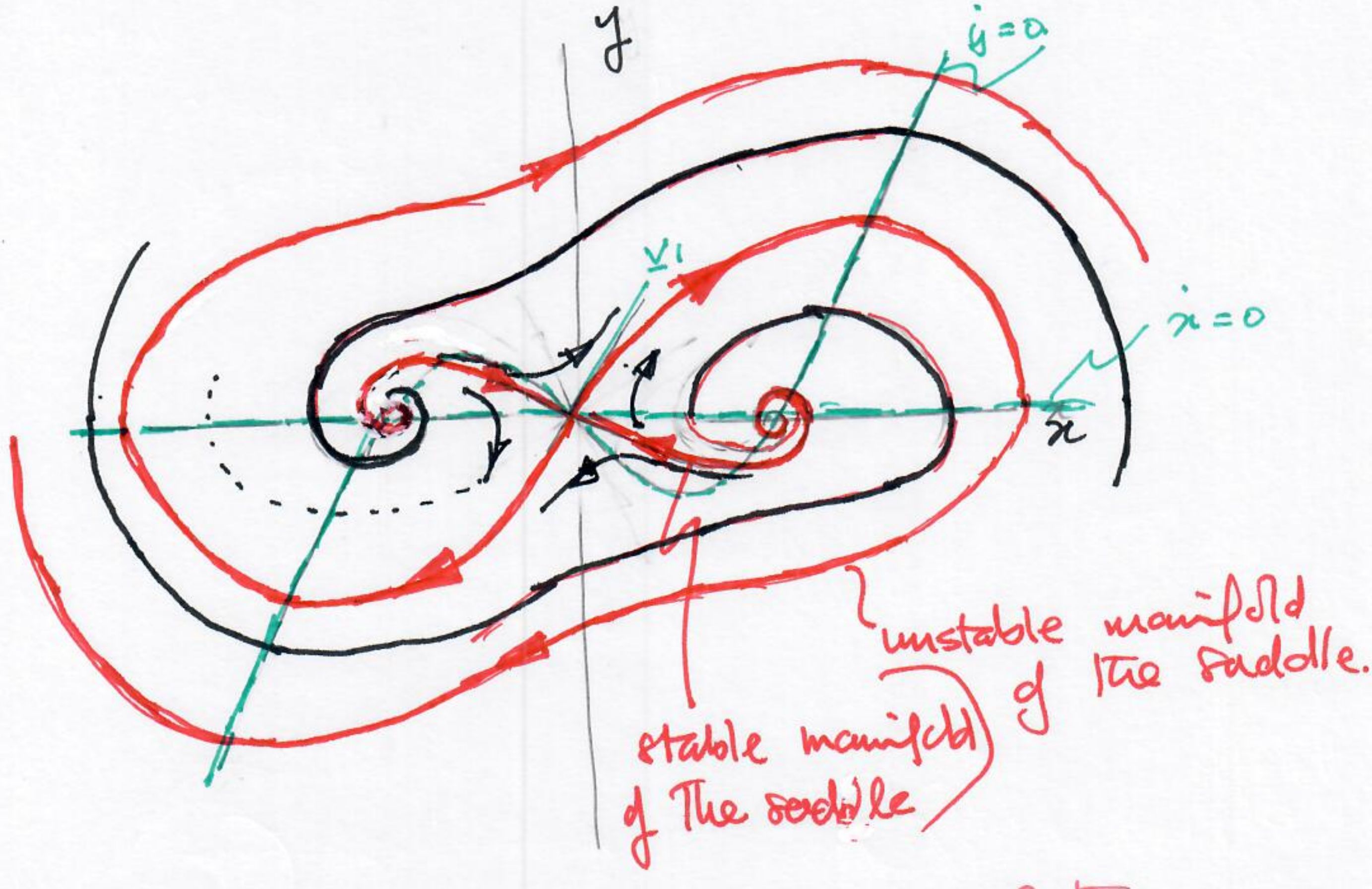
(c)(i) $\dot{x} = 0 \Rightarrow y = 0$ (vertical)

$\dot{y} = 0 \Rightarrow y = x^3 - x$ (horizontal)

(C)
(ii & iii)



(d)



seen technique but not the system.