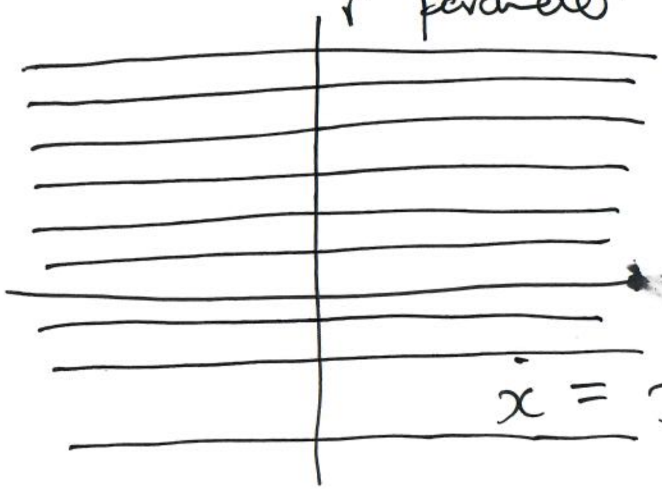
Tutorial 11 in Week 12

T12.1

General chat about introduction

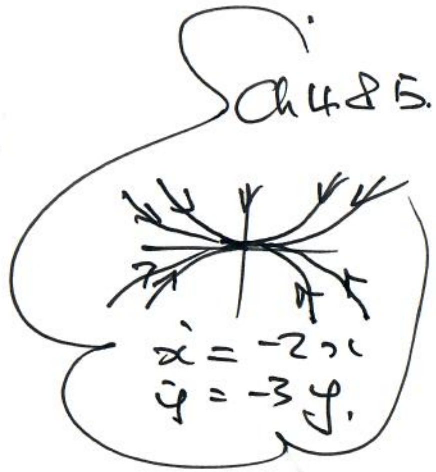
$$\begin{cases} \dot{x} = f(x,r) = rx - x^2 + x^4. \\ \dot{r} = 0. \end{cases}$$

Q2 bifurcation question.



$\dot{x} = f(x, r)$

cf.



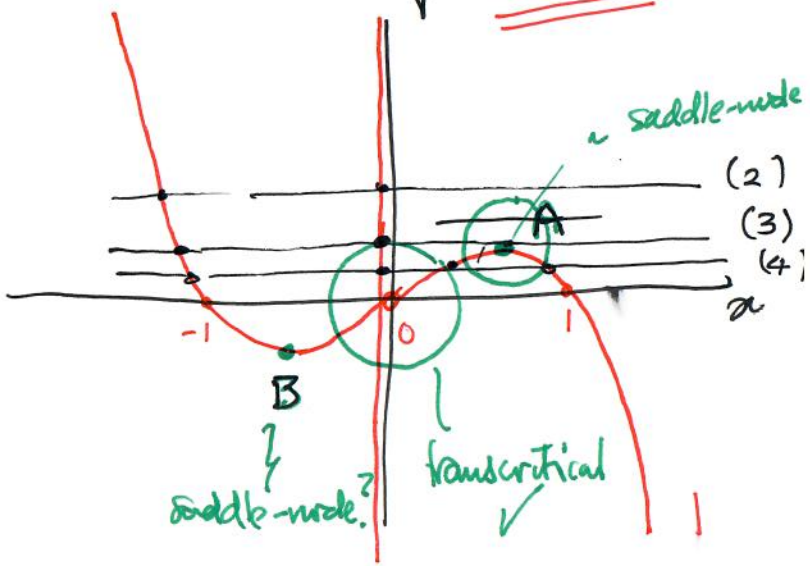
$\dot{x} = -2x$
$\dot{y} = -3y.$

$$\dot{x} = rx - x^2 + x^4 = x(r - x + x^3)$$

FPs $\quad x = 0, \quad r = x - x^3$
$= x(1 - x^2)$

$r = 0$ when $x = 0,\ x = \pm 1$

FPs



(a) $\dot{x} = rx - x^2 + x^4 = f(x,r)$ $x = 0$. $\dot{x} = 0$ $\forall r$. $\boxed{T12.2}$

Linear stability $\frac{\partial}{\partial x}(f(x,r)) < 0$ lin stable.

> 0 lin unstable

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = r - 2x + 4x^3\Big|_{x=0} = r$$

$r > 0$ unstable
$r < 0$ stable $\Big\}$ $r = 0$? change of stability

bifurcation at $r = 0$

$r = 0$ is the parameter value.

$x = 0, r = 0$ $f(x,r)$ is already a Taylor expansion centred on

$x=0, r=0$ $f(x,r)$ is polynomial.

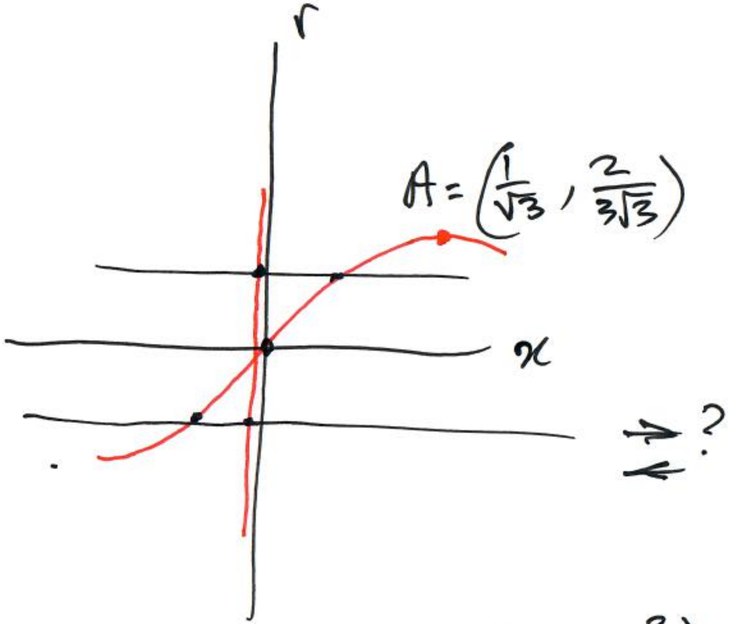
"A", "B", "C" $\dot{x} = rx - x^2 + x^4$ $A = 0$ $C = 1$, $B = -1$

$\Downarrow$

coeff of r x^2 xr transcritical bifurcation

μ y^2 $y\mu$ – (local coord)

[T12.3]



Conjecture A, B are bifurcation points for a saddle-node bifurcation

Coordinate of A (local max of $y = x - x^3$.)

$$\frac{dy}{dx} = 1 - 3x^2.$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}.$$

$$\frac{d^2y}{dx^2} = -6x \quad \begin{array}{ll} \text{Max} < 0 & x > 0 \\ \text{Min} > 0 & x < 0 \end{array}$$

Max $(\cancel{x} - x^3)$ is given by $x = \frac{1}{\sqrt{3}}$

and $\cancel{y} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$

xr diagram

$$A = \left(\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$$

$$B = \left(-\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right)$$

$$\text{Let } \left.\begin{array}{l} y = x - \frac{1}{\sqrt{3}} \\ \mu = r - 2/3\sqrt{3} \end{array}\right\} \text{ pt } A$$

T2.4

$$\dot{y} = \dot{x} = xr - x^2 + x^4 = \left(y+\frac{1}{\sqrt{3}}\right)\left(\left(\mu+\frac{2}{3\sqrt{3}}\right) - \left(y+\frac{1}{\sqrt{3}}\right)^2 + \left(y+\frac{1}{\sqrt{3}}\right)^3\right)$$

$$= \left(y+\frac{1}{\sqrt{3}}\right)\left[\mu+\frac{2}{3\sqrt{3}} - y - \frac{1}{\sqrt{3}} + y^3 + 3\frac{1}{\sqrt{3}}y^2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 y + \left(\frac{1}{\sqrt{3}}\right)^3\right]$$

$$= \left(y+\frac{1}{\sqrt{3}}\right)\left[\mu+\frac{2}{3\sqrt{3}} - y - \frac{1}{\sqrt{3}} + y^3 + \sqrt{3}y^2 + y + \frac{1}{3\sqrt{3}}\right]$$

$$= \left(y+\frac{1}{\sqrt{3}}\right)\left[\mu + y^3 + \sqrt{3}y^2\right]$$

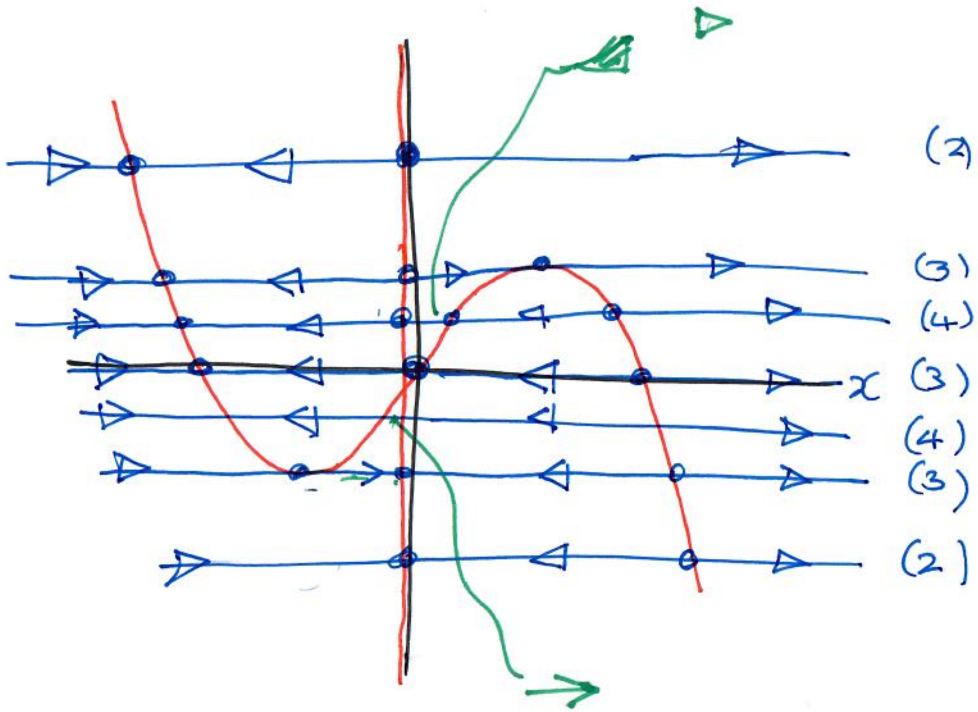
$$= \underbrace{\boxed{\mu y}}_{=C} + \underbrace{\boxed{\frac{1}{\sqrt{3}}\mu}}_{=A} + y^4 + \frac{1}{\sqrt{3}}y^3 + \sqrt{3}y^3 + \underbrace{\boxed{y^2}}_{=B}.$$

Coeff C A B

Can be anything if $A, B \neq 0$.

$A \neq 0$, $B \neq 0$, $C \neq 0$ saddle node

T12.5



orientation of the orbits of the various phase portraits?

$$\dot{x} = rx - x^2 + x^4$$

T12.6

Q3

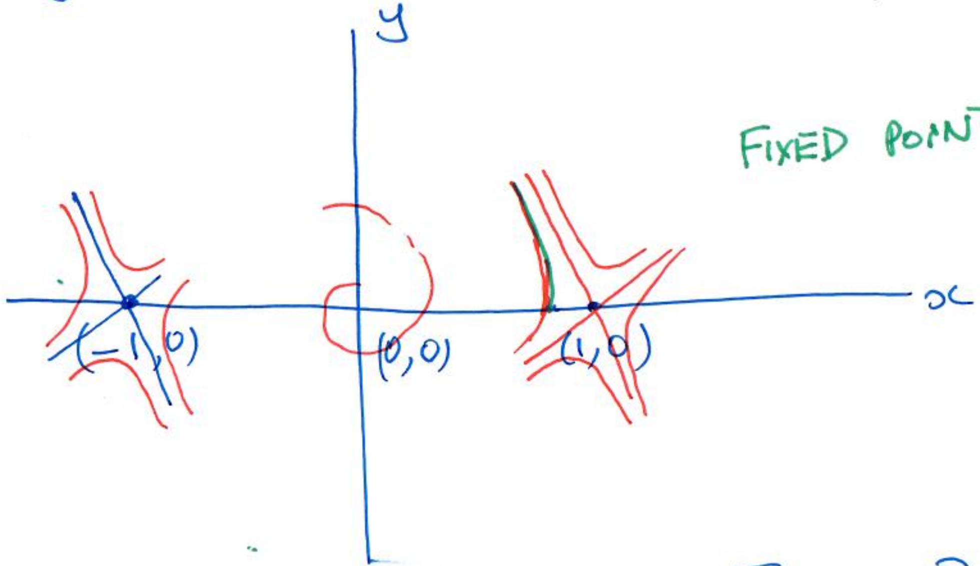
$\dot{x} = y \quad , \quad \dot{y} = -x + x^3 - y$

$\dot{x} = 0 \ : \ y = 0$

$\dot{y} = 0 \ : \ -x + x^3 = 0$

$x = \pm 1, 0.$

FIXED POINTS



$$J_{(x,y)} = \begin{bmatrix} 0 & 1 \\ -1+3x^2 & -1 \end{bmatrix}_{(x,y)}$$

$$J(-1,0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad J(+1,0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Eigenvalues

$(0,0)$ $\quad \lambda(\lambda+1) + 1 = 0 \quad \lambda^2 + \lambda + 1 = 0 \quad \lambda = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1}{2} \pm \sqrt{\dfrac{3}{2}}\, i$

stable spiral $\quad \alpha = \dfrac{-1}{2}$ (re part < 0)

$(-1,0)$ $\quad \lambda(\lambda+1) - 2 = 0 \quad \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) = 0 \quad \lambda_1 = -2, \ \lambda_2 = 1$

T12.7

<u>Eigenvectors</u>

$\lambda_1 = -2 - \quad \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = -2\begin{bmatrix} u \\ v \end{bmatrix} \qquad v = -2u \qquad \underline{v}_1 = (1, -2)$

$\lambda_1 = 1 + \quad \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 1\begin{bmatrix} u \\ v \end{bmatrix} \qquad v = u \qquad v_2 = (1, 1)$

Same for $(-1, 0)$ and $(1, 0)$

(0,0)

(−1,0)
(+1,0)

T12.8

$$\dot{x} = y$$
$$\dot{y} = -x + x^3 - y$$