TUTORIAL 8 18.1

Some examples from Chap 5 (Note: mp4 shortly available in Video tab on QMPlus)

Ex 5.6 $\dot{x} = y, \quad \dot{y} = x - x^3$

FPs $y = 0, \quad x - x^3 = 0 \quad \Rightarrow \quad x = -1, 0, +1$

$(-1,0), (0,0), (1,0)$

$\dot{x} = f(x,y)$

$\dot{y} = g(x,y)$

Jacobian matrix $Df(\underline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$

$f(x,y) = y$

$g(x,y) = x - x^3$

$\rightarrow \quad Df(\underline{x}) = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$

$\underline{x} = (-1,0) \quad Df(\underline{x}) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\quad Tr = 0, \quad Det = 2$

$\lambda_{1,2} = \frac{0 \pm \sqrt{-8}}{2} = \pm\sqrt{2}\, i$ ~~spirals~~ centre type linearly

$\rightarrow$? centre ~~type~~ non-linearly HGLT(?).

$\underline{x} = (1,0)$ $\quad \shortparallel$

$\underline{x} = (0,0) \quad Df(\underline{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda_{1,2} = \pm 1 \quad$ saddle

8.2

$\lambda_1 = 1$ unstable eigenvector

$\lambda_2 = -1$ stable eigenvector.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 1\begin{bmatrix} u \\ v \end{bmatrix}$ $v = u$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = -1\begin{bmatrix} u \\ v \end{bmatrix}$ $v = -u$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

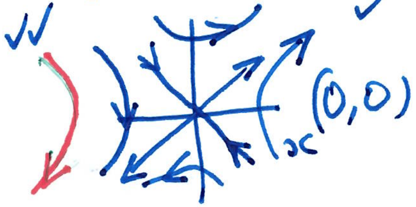
linearly ~~stab~~ centre type

$\dot{x} = y$

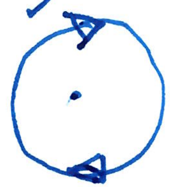
(⇒ clockwise)

$\dot{x} > 0$ for $y > 0$

$\dot{x} > 0$



Linear



Circles or spirals

Non-hyperbolic?

stays as a saddle

hyperbolic ✓

non-hyperbolic?

Eliminating t gives

$\dot{x} = y, \quad \dot{y} = x - x^3$

$$\frac{dy}{dx} = \frac{x - x^3}{y} \Rightarrow y\,dy + (x^3 - x)\,dx = 0$$

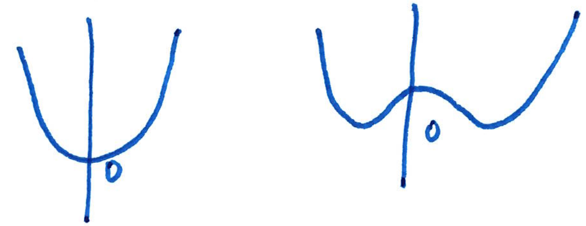
$$\frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2} = \text{Const.}$$

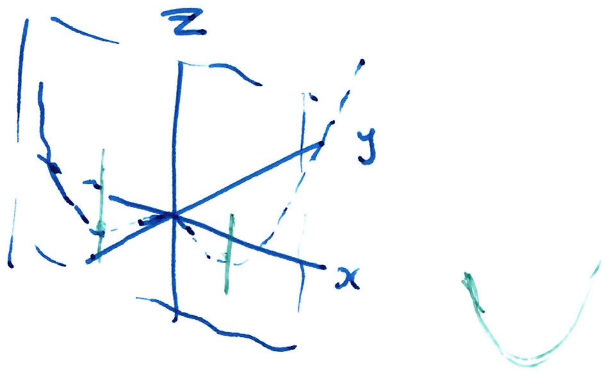
– this is a conserved quantity for orbits of the dynamical system.

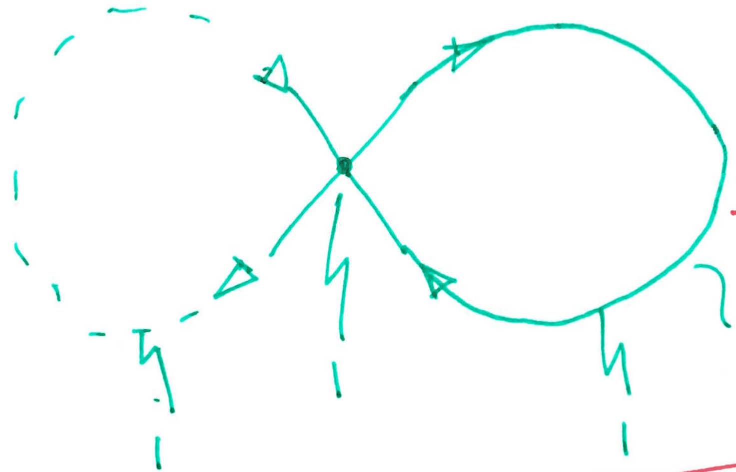
$\boxed{8.3}$

$$z = V(x,y) = \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}.$$

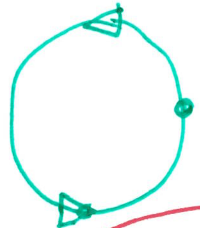
fn of y fn of x







saddle



homoclinic connection

$\dot{\theta} = 1 - \cos(\theta)$

Conserved value $V(x,y)$

orbits (soln curves) lie on $V = \text{Constant}$

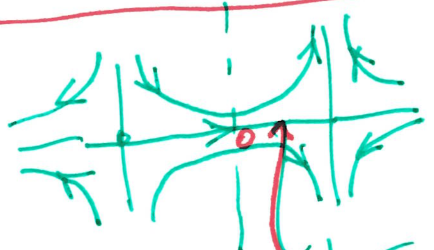
$y = -y$??? $\dot{x} = -x^2 + 1$

Second try

$\dot{x} = -x^2 + 1$

$\dot{y} = x * y$

see attached



$y=0$

$x=0$

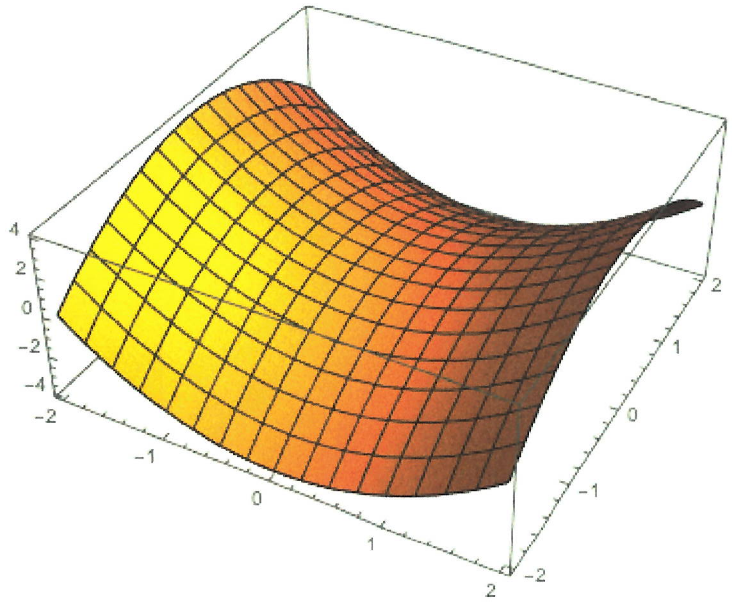
heteroclinic saddle (connecting two distinct saddles)

Similar for $\dot{y} > 0$ $y>0$, $x>0$ and $y<0$, $x<0$

Note $\dot{y} < 0$ for $y>0$, $x<0$ and $y<0$, $x>0$

In[16]:= `Plot3D[x^2 - y^2, {x, -2, 2}, {y, -2, 2}]`

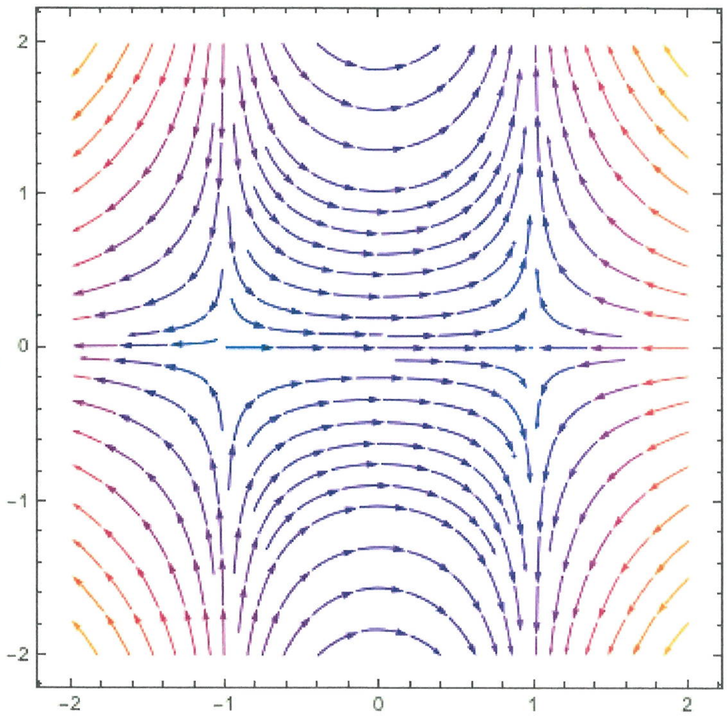
Out[16]=



In[22]:=

```
StreamPlot[{1 - x^2, x*y}, {x, -2, 2}, {y, -2, 2}]
```

Out[22]=



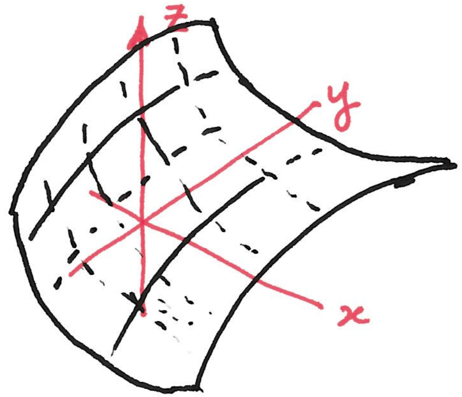
$\boxed{8.5}$

What are the level curves of (i) $V(x,y) = x^2 + y^2$

(ii) $V(x,y) = x^4 + y^4$

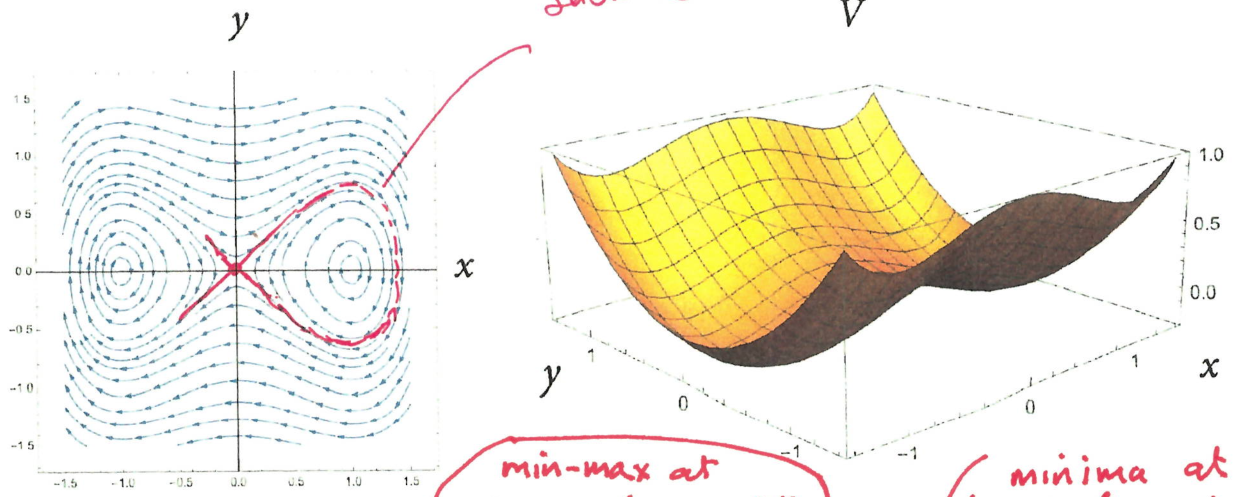
(iii) $V(x,y) = x^2 - y^2$

(iii) $z = V(x,y)$



(i) & (ii)

saddle connection



(a) (b)

min-max at (0,0) gives saddle
Contour = 0 gives a double saddle connection

minima at $(x,y) = (\pm 1, 0)$ give closed orbits

Figure 21: *(a) The phase portrait for the system 5.23 follows the level curves of the first integral V. (b) Note the 'Mathematica' picture of the surface $z = V(x,y)$ does not capture the saddle point at the origin and its unstable/stable manifolds, but it does show up the non-linear centres at $(\pm 1, 0)$ well.*

The critical(stationary) points of the surface $z = V(x,y)$ occur when the condition $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$. These are the fixed points of the underlying system. The type of critical point - maximum, minimum, saddle is determined by the discriminant

$$D = V_{xx}V_{yy} - V_{xy}^2,$$

which gives: maximum for $D > 0$ and $V_{xx} < 0$; minimum for $D > 0$ and $V_{xx} > 0$; saddle for $D < 0$.

So the critical points are $(0,0)$ - saddle and $(1,0)$, $(-1,0)$ - both minimums. Thus this means that the orbits around the fixed points $(1,0)$, $(-1,0)$ are closed (periodic orbits), and the "saddle point" or "col" of the surface confirms the existence of the saddle fixed point by the HGLT. It should be noted that the contours show the global nature of the orbits of the system. We see that the unstable and stable saddle manifolds coincide, i.e. the saddle unstable manifolds leave the fixed point $(0,0)$ and fold around to return as the stable manifolds! The contour through the saddle point has the form of a "figure-8". This is a highly non-linear feature which contrasts with the straight lines of a linear saddle.It is called a pair of saddle-connections as the unstable manifold and the stable manifolds of the saddle are the same in each of the two branches!

Example 5.7.

$$\mathbf{f}(\mathbf{x}) = (x(3 - (x + 2y)), y(2 - (x + y))) \tag{5.18}$$

The Lotka-Volterra model of competing species. The simplest model for each species with different reproductive rates and carrying capacity could be modelled by the decoupled logistic equations

$$\mathbf{f}(\mathbf{x}) = (x(3 - x), y(2 - y)). \tag{5.19}$$

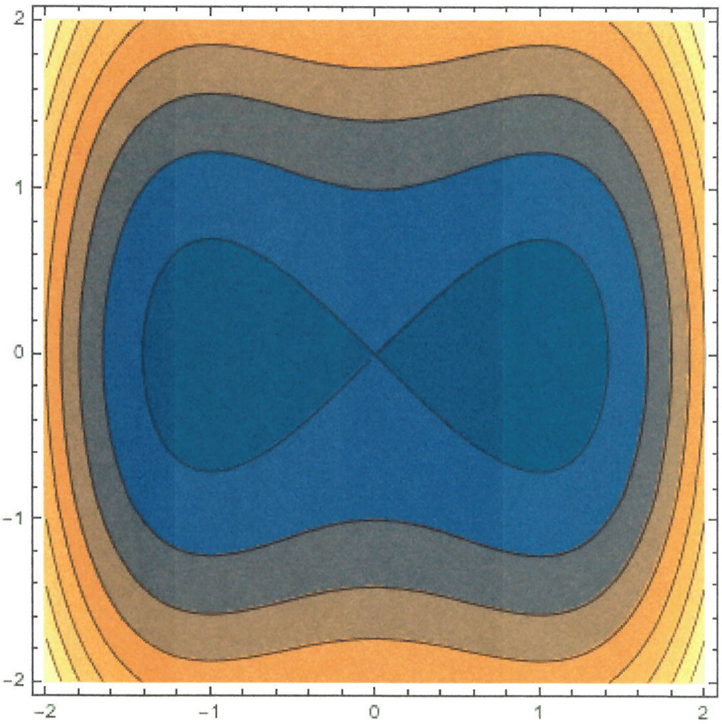
If coupling terms are introduced to model the interaction between the two species, then we obtain

$$\dot{x} = (x(3 - (x + 2y)), \ \dot{y} = y(2 - (x + y)) \tag{5.20}$$

In[12]:=

```
ContourPlot[x^4/4 - x^2/2 + y^2/2, {x, -2, 2}, {y, -2, 2}]
```

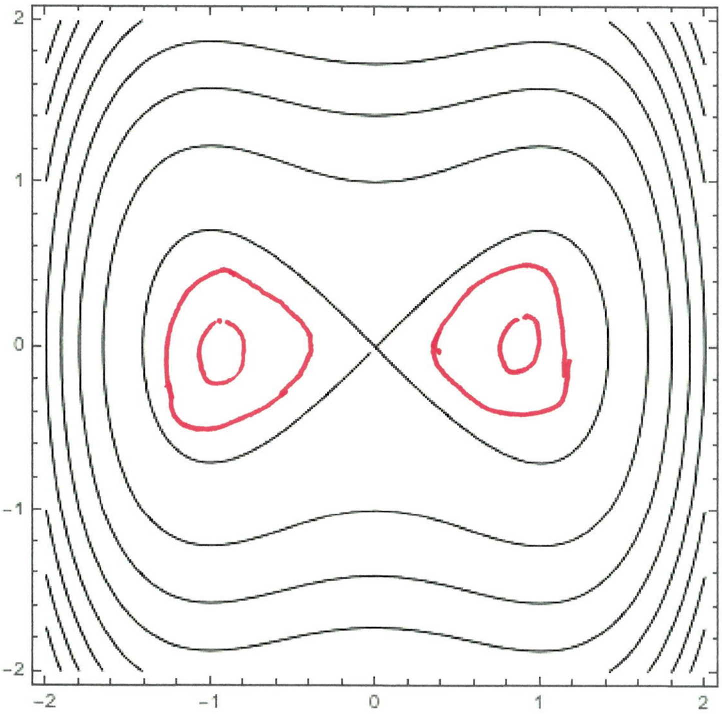
Out[12]=



These are the contours corresponding to Figure 21

In[13]:= ContourPlot$\left[\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}, \{x, -2, 2\}, \{y, -2, 2\}, \text{ContourShading} \to \text{None}\right]$

Out[13]=



8.8

$\dot{x} = x(3-(x+2y))$ $\dot{y} = y(2-(x+y))$

FPs $(x=0$ or $3-(x+2y)=0)$ **and** $y=0$ or $2-(x+y)=0$

$x=0, \ y=0$ $(0,0)$

$x=0, \ 2-(x+y)=0$ $(0,2)$

$3-(x+2y)=0, \ y=0$ $(3,0)$

$3-(x+2y)=0$ $2-(x+y)=0$ $(1,1)$

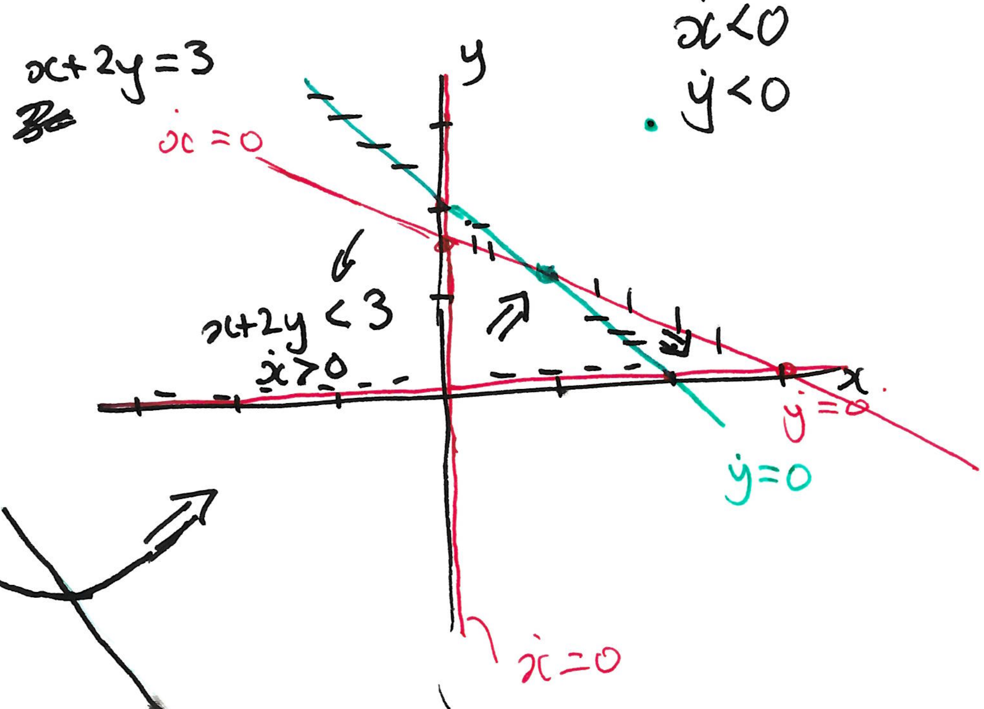
$x+2y>3$

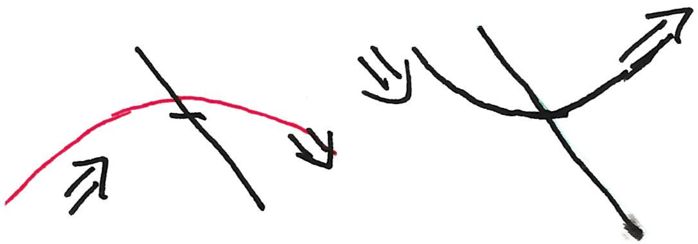
Linearisation.

Nullclines $\dot{x}=0, \ \dot{y}=0$

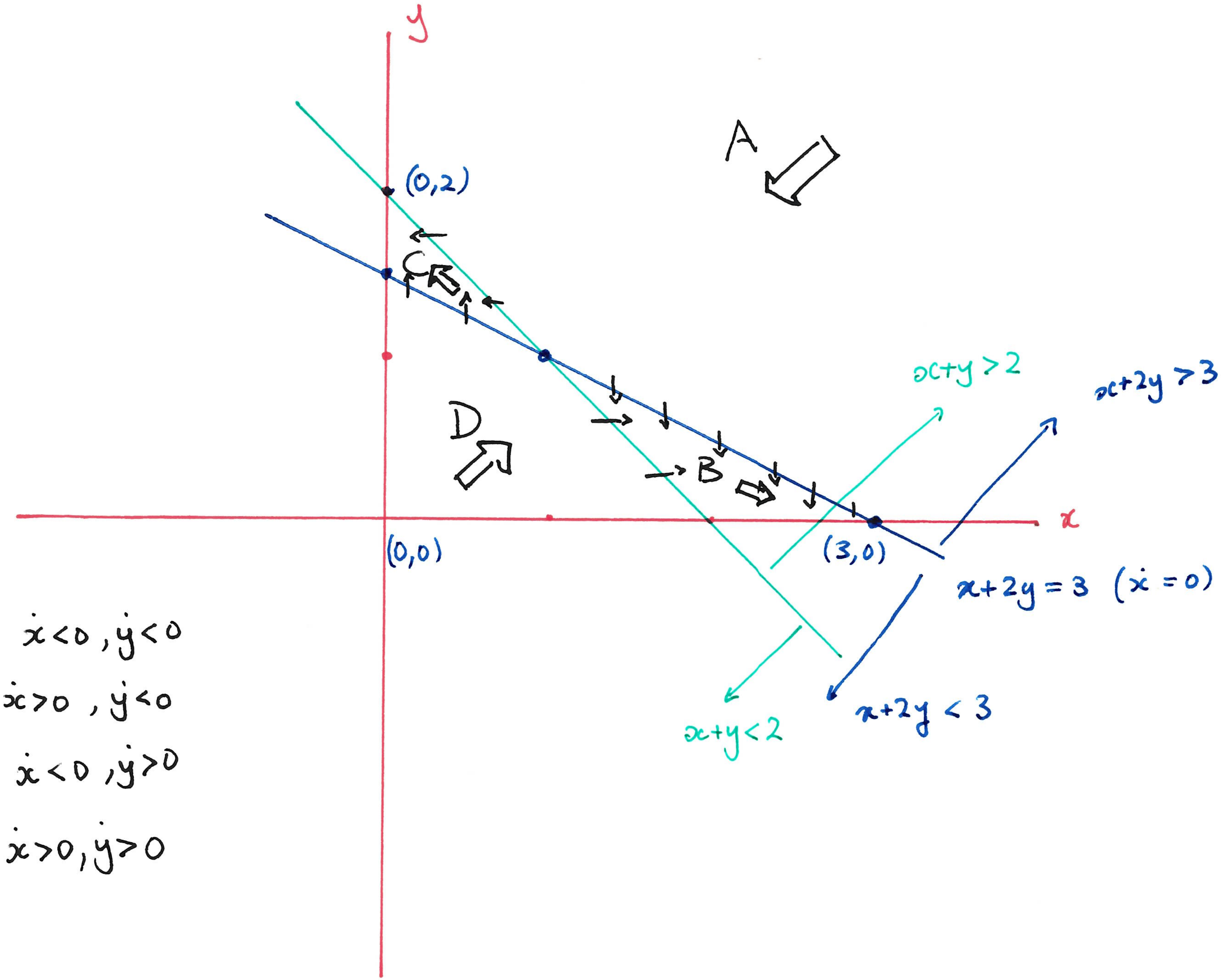
$\dot{x}=0$ $x=0, \ x+2y=3$

$\dot{y}=0$ $y=0, \ x+y=2$





8.9



Regions
A $\dot{x}<0, \dot{y}<0$
B $\dot{x}>0, \dot{y}<0$
C $\dot{x}<0, \dot{y}>0$
D $\dot{x}>0, \dot{y}>0$