Tutorial 7 (wk 8)

Linear systems in $\mathbb{R}^2$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

→ eigenvalues help determine the nature of the fixed

λ_1, λ_2, $\lambda_1 \neq \lambda_2$ $\underline{v}_1, \underline{v}_2$ - eigenvectors.

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \quad \text{Det} = 6 \quad \text{Tr} = 5.$$

$\lambda^2 - \text{Tr}(A)\lambda + \text{Det}(A)$ $\qquad \lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 2, \ \lambda_2 = 3$ $\qquad$ EV equ $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 2 \begin{pmatrix} u \\ v \end{pmatrix}$

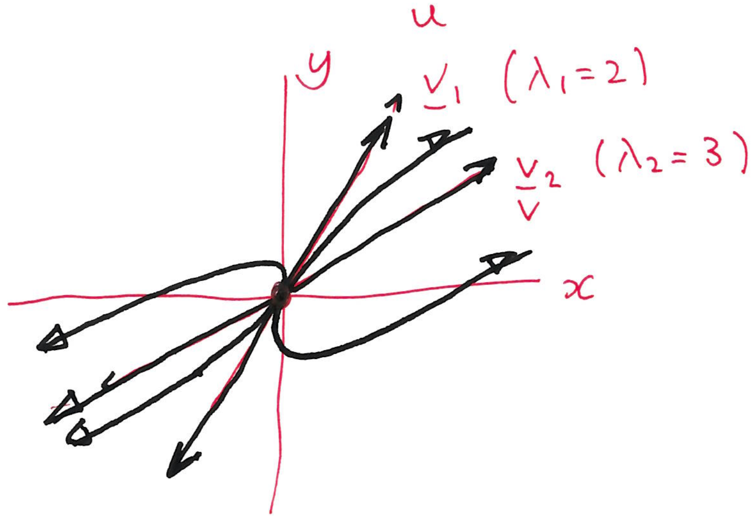
$\lambda_1 = 2$ $\quad \left.\begin{array}{l} u + 2v = 2u \\ -u + 4v = 2v \end{array}\right\} \Rightarrow u = 2v$ $\qquad \underline{v}_1 = (2, 1)^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$ $\quad \left.\begin{array}{l} u + 2v = 3u \\ -u + 4v = 3v \end{array}\right\} \Rightarrow u = v$ $\qquad \underline{v}_2 = [1, 1] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$A \underline{v} = \lambda \underline{v}$

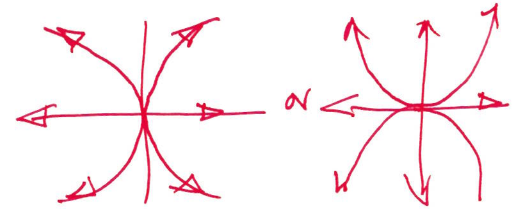
col. vector

2



Fixed pt $\lambda_1 = 2, \lambda_2 = 3$

unstable node



$\dot{x} = \lambda_1 x$

$\dot{y} = \lambda_2 y$

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 2^{\lambda_1} & 0 \\ 0 & 3^{\lambda_2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

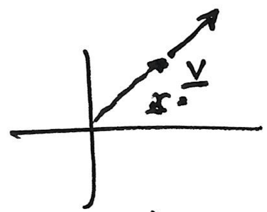
$$\frac{dv}{du} = \left(\frac{\lambda_2}{\lambda_1}\right) \frac{v}{u} \qquad \frac{dv}{v} = \frac{\lambda_2}{\lambda_1} \frac{du}{u} \Rightarrow v = C u^{(\lambda_2/\lambda_1)} = C u^{3/2}$$

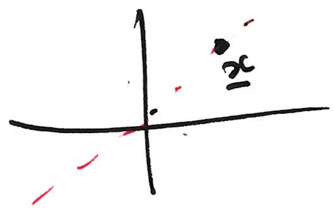
tangent to u

$A\underline{x} = \lambda x$

$\underline{\dot{x}} = A\underline{x}$ $\qquad \underline{x} = \underline{v}$ - eigenvector

$A\underline{v} = \lambda \underline{v}$





eigendirection - they are invariant curves - trajectories

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \underline{\dot{x}} = \underline{A}\,\underline{x}$$

$$\mathrm{Det}(A) = -1 \qquad \mathrm{Tr}(A) = 2 \qquad \lambda^2 - 2\lambda - 1$$

$$\lambda = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

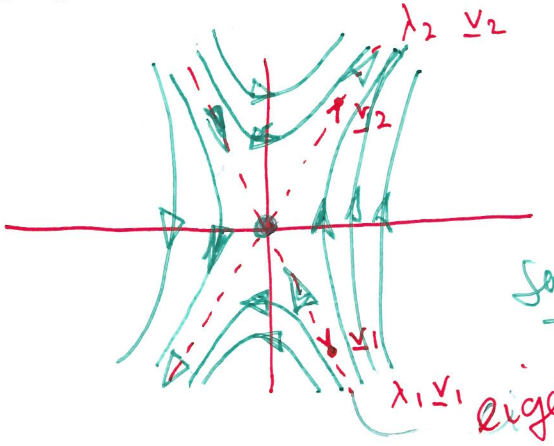
$$\lambda_1 = 1-\sqrt{2}, \qquad \lambda_2 = 1+\sqrt{2}$$

$$\underline{v}_1 = \begin{pmatrix} u \\ v \end{pmatrix} \qquad A\underline{v}_1 = \lambda_1 \underline{v}_1 \Rightarrow \begin{aligned} u+v &= (1-\sqrt{2})u \Rightarrow \sqrt{2}u = -v \\ 2u+v &= (1-\sqrt{2})v \Rightarrow 2u = -\sqrt{2}v \\ &\qquad \sqrt{2}u = -v \ \checkmark \end{aligned}$$

neg $\lambda_1 = 1-\sqrt{2}, \quad \underline{v}_1 = (\qquad)$

pos $\lambda_2 = 1+\sqrt{2}, \quad \underline{v}_2 = (1, \sqrt{2}) \qquad u+v = (1+\sqrt{2})u \qquad v = \sqrt{2}u$

$v = C u^{\lambda_2/\lambda_1}$ (u, v coords relative to $\underline{v}_1$ & $\underline{v}_2$)



saddle

eigen direction has 3 trajectories

$\dot{u} = \lambda_1 u \qquad \dot{u} = (1-\sqrt{2})u$

$\dot{v} = \lambda_2 v$

$v = 0$

$\frac{\lambda_2}{\lambda_1} < 0$

$y = C x^{\lambda_2/\lambda_1}$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \overset{A}{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix} \qquad \lambda_1 = 1 \;,\; \lambda_2 = 1$$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix}$$

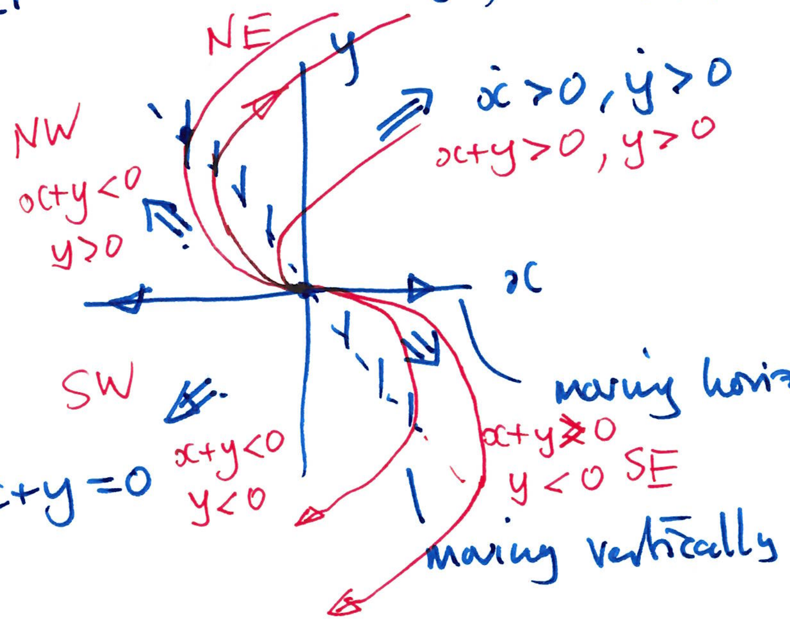
$$A\,\underline{v}_1 = \lambda\,\underline{v}_1 \qquad \underline{v}_1 = \begin{pmatrix}u\\ v\end{pmatrix} \qquad \begin{aligned} u + v &= u\\ 0 + v &= v\end{aligned}$$

$v = 0$

$\underline{v}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$

$\dot{x} = x$

$y = 0$



$\underline{\dot{x} = 0}$? $\quad x + y = 0$

null-cline

$\dot{y} = 0 \qquad y = 0$

$\dot{x}, \dot{y} > 0$ NE

$\dot{x} < 0 \;\; \dot{y} > 0$ NW

$\dot{x} < 0 \;\; \dot{y} < 0$ SW

$\dot{x} > 0 \;\; \dot{y} < 0$ SE