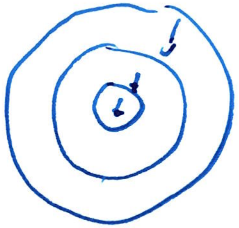
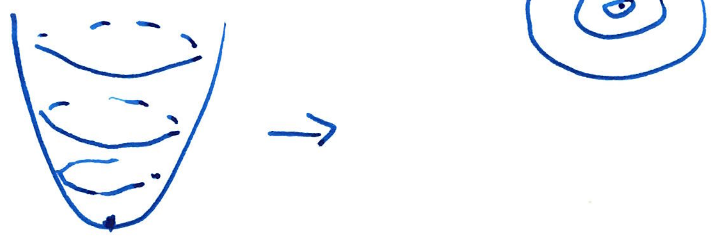
Liapounov Stability

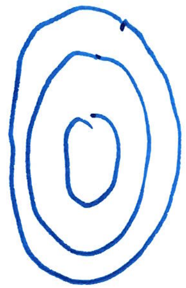
$\dot{r} < 0$

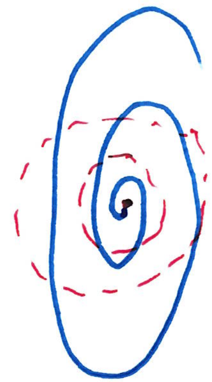


stability / attraction to the fixed point

→ we extend the set of functions which can be used to show stability by reducing the distance of the orbit from the fixed point as t increases.







19.2

5.5 $\dot{x} = y - x r^2, \quad \dot{y} = -x - y r^2$

$\dot{x} = y, \quad \dot{y} = -x, \quad r, \theta$

$r\dot{r} = -(x^2+y^2) r^2 = -r^4 \Rightarrow \dot{r} = -r^3.$

$r(t) = \sqrt{x(t)^2 + y(t)^2} \to 0$ as $t \to \infty$

$(x(t), y(t))$ is a soln of the system.

$L(x,y) = x^2 + y^2$

$\dot{L}(x,y) = -r^4 \; (= -(x^2+y^2)^2.$ $\quad \frac{dL}{dt} < 0$

L decreases with time → distance of $\underline{x}(t)$ ~~der~~ from $\underline{0}$ decreases with time.

So we want to some calculations with the same conclusions for more general functions than $L(x,y) = x^2+y^2$ (polar distance)

Ex 5.10 (ii) $\ddot{x} + \dot{x}^3 + x = 0$

Let $\dot{x} = y, \quad \dot{y} = \ddot{x} = -\dot{x}^3 - x = -y^3 - x$

$\dot{x} = y, \quad \dot{y} = -y^3 - x$

If this was a gradient system

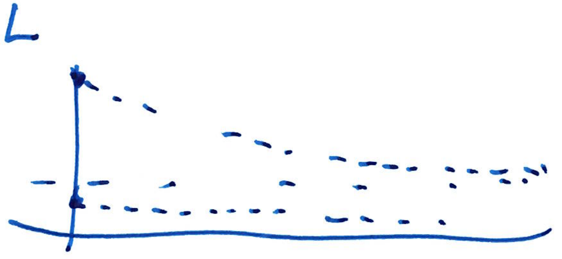
$\dot{x} = \dfrac{-\partial F}{\partial x}, \quad \dot{y} = \dfrac{-\partial F}{\partial y}$, for some F.

$-\dfrac{\partial F}{\partial x} = y \Rightarrow, \dfrac{-\partial^2 F}{\partial x \partial y} = 1$ $\quad \#$ $\quad \therefore$ F does not $\exists$.

$-\dfrac{\partial F}{\partial y} = -y^3 - x \quad -\dfrac{\partial^2 F}{\partial y \partial x} = -1$ $\quad \therefore$ Non-gradient

$\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x} \iff$ F exist for the gradient.

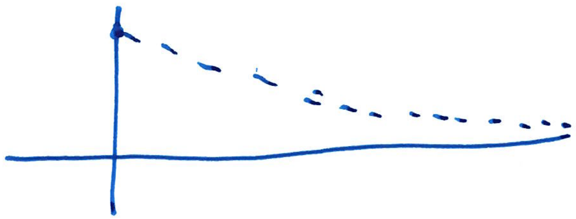
$\frac{dL}{dt} \leq 0$ L



$\rightarrow L_0$ as $t \to \infty$

$\frac{dL}{dt} < 0$

strict L



$\rightarrow L = 0$. as $t \to \infty$

<u>Ex 5.11</u> Construct a Liapunov function for

$$\dot{x} = -x + 4y, \quad \dot{y} = \cancel{-x} - y^3$$

try with b=0

Let $L(x,y) = ax^2 + \cancel{bxy} + cy^2$. - quadratic functional.

$$\dot{L} = \cancel{2ax}\ \cancel{\dot{x}}$$

$$\frac{\partial L}{\partial x} \cdot \dot{x} + \frac{\partial L}{\partial y} \dot{y} = (2ax + \cancel{by})\dot{x} + (\cancel{bx} + 2cy)\dot{y}$$

Note "a" and "c" used differently from the Lecture note version

19.5

$$= (2ax + \cancel{by})(-x + 4y) + (2cy + \cancel{bx})(-x - y^3)$$

$$= -2ax^2 + 8axy - 2cxy - 2cy^4$$

$$= -2ax^2 + 2xy(4a - c) - 2cy^4$$

Choose $4a = c$: $\Rightarrow = -2ax^2 - 2cy^4$

$a, c > 0$

$$L(x,y) = ax^2 + cy^2. \quad \text{PD?}$$

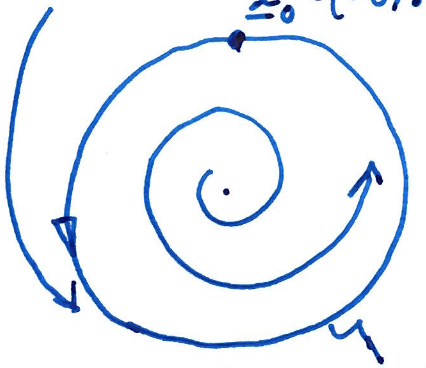
$$\frac{dL}{dt}(x,y) = -2ax^2 - 2cy^4 \quad \text{ND?}$$

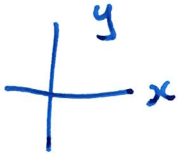
$a=1$, $c=4$

$\therefore$ Asymptotic stability

5.6. Limit cycle is a completely non-linear phenomenon.

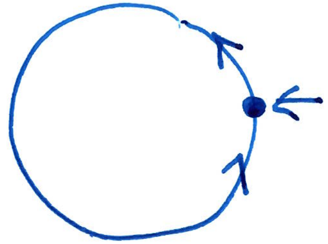
$\underline{x}_0 = (x_0, y_0)$





autonomous system

$\dot{\theta} = 1 - \cos\theta.$



$\underline{x}(t) = (x(t), y(t))$

Periodic returns every time advance of T

$\underline{x}(0) = \underline{x}_0$

$2T$ ✓

$\underline{x}(T) = \underline{x}_0$

$\underline{x}(2T) = \underline{x}_0$

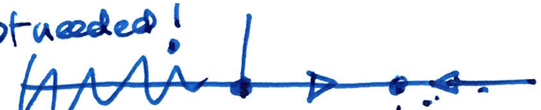
Period? minimum positive T

"first return time"

Ex 5.12

$$\dot r = r(1-r)\ , \quad \dot\theta = 1 \quad \rightarrow$$

not needed!



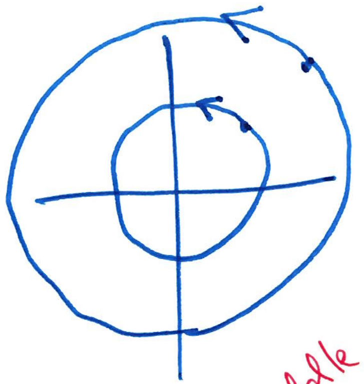
polar coordinate

fixed point.

$$\dot r = r(1-r)^2 \quad \dot\theta = 1$$

$\dot x = y$

$\dot y = -x$



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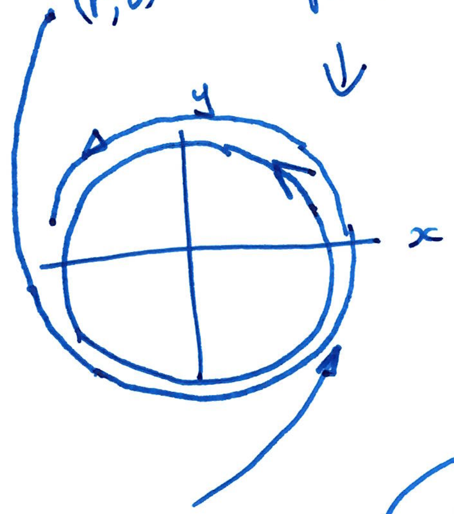
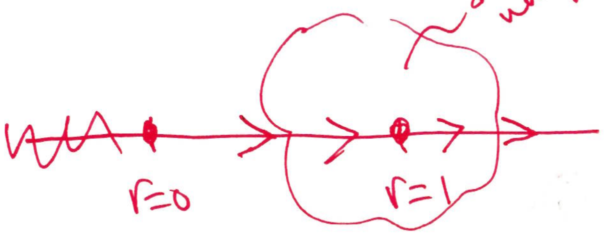
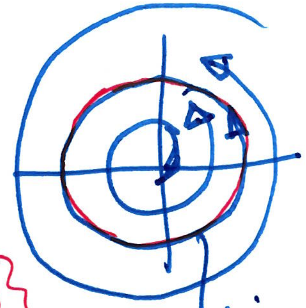


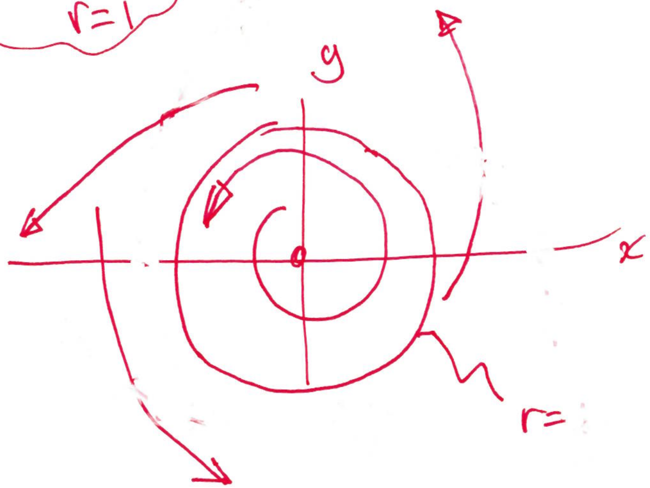
saddle node



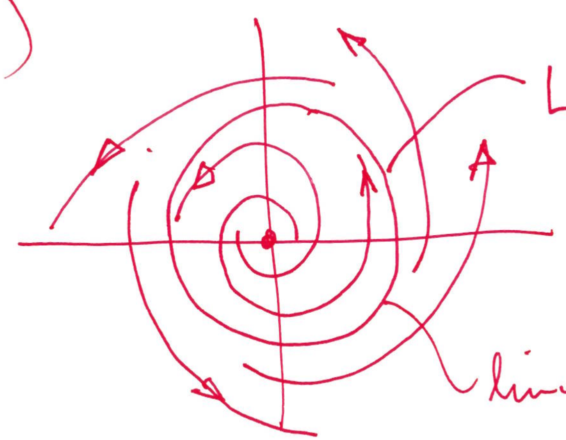
Basin of attraction of the cycle is $r \equiv \mathbb{R}^2 - \{0\}$



limit cycle



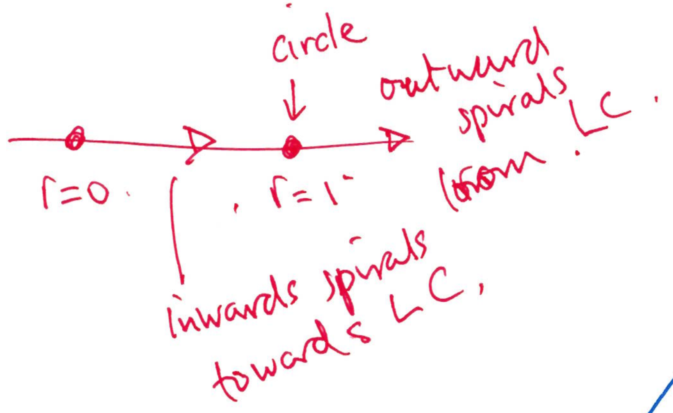
(ii)

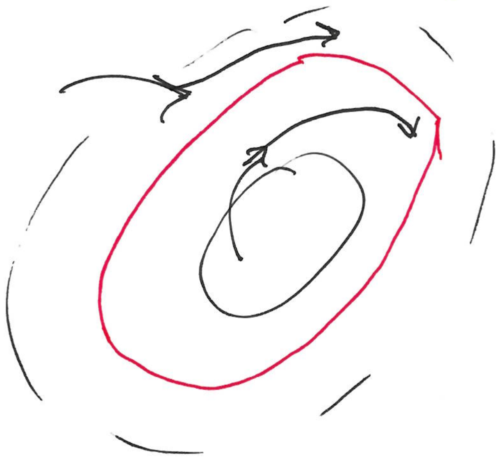


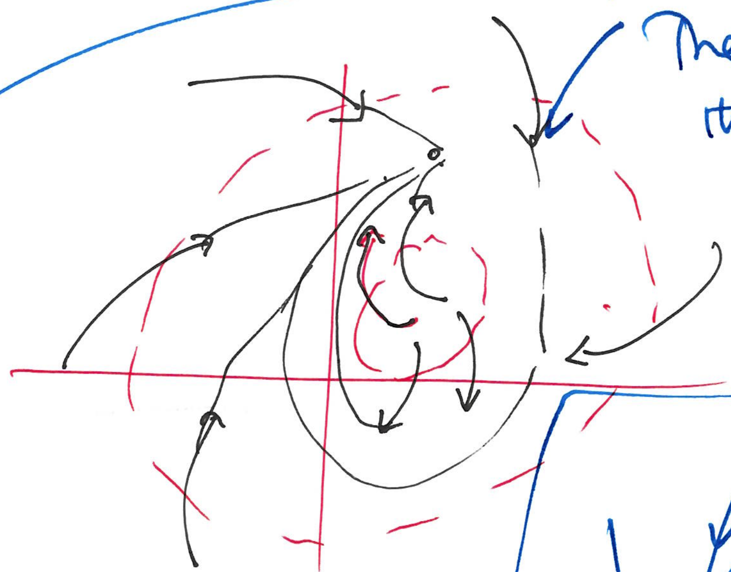
stable from inside
unstable from the outside
"unstable"

→ a saddle-node in the radial direction

$+\dot{\theta}=1$

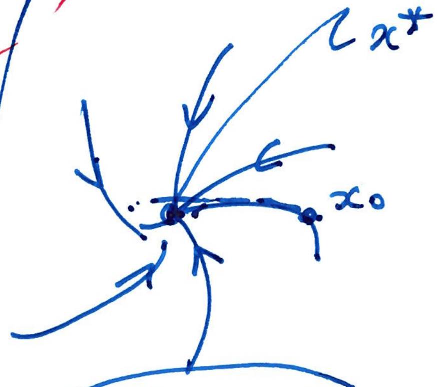




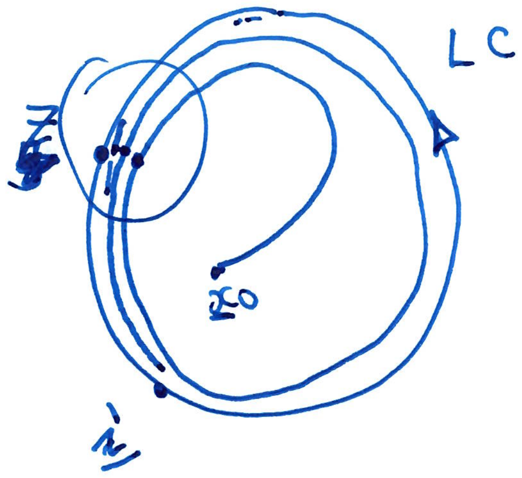


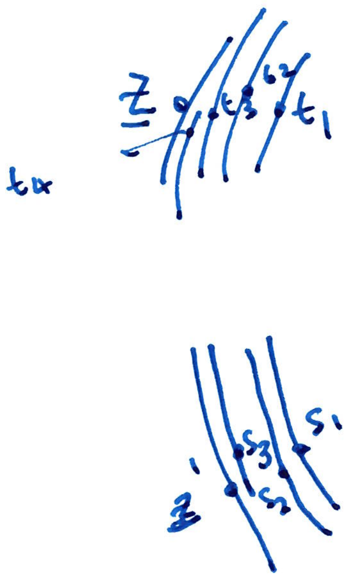
Poincaré-Bendixson Theorem predicts the existence of periodic orbit(s) in a trapping region.

provided it is fixed point free!



w-limit set
x^* is w-limit pt of x_0





$\underline{x}(t_n) \to \underline{z}$

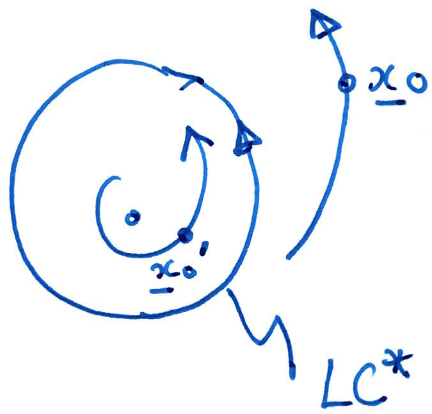
$\underline{z} \in \omega(\underline{x}_0)$

$\underline{x}(s_n) \to \underline{z}'$

$\omega(\underline{x}_0) = LC.$

α-limit set

Ex (ii)



$\alpha(\underline{x}_0) = LC^*$

$t_n \to -\infty$

$\omega(\underline{x}_0') = LC^*$

$t_n \to +\infty$