Lectures 17–18 (Please note considerable reference to lecture notes during lectures on 29th Nov – notes ∴ scrappy) $\boxed{17.1}$

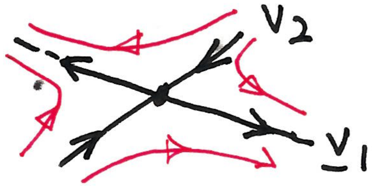
RESUME / RECALL

Saddle

$\lambda_1, \lambda_2 \neq 0$ $\qquad \lambda_1 \lambda_2 < 0$ $\qquad \lambda_1 > 0, \lambda_2 < 0$

$\updownarrow \qquad \updownarrow$

$\underline{v}_1, \underline{v}_2$

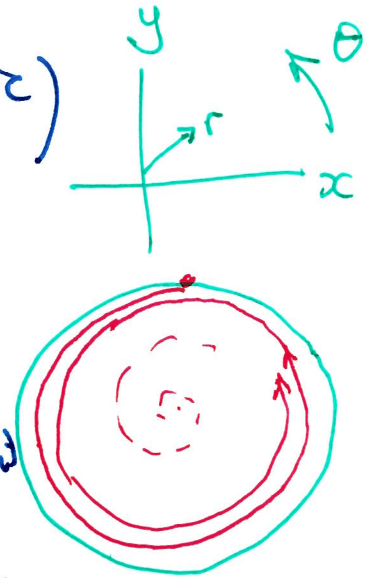


eigenvectors critical to shape of saddle.

Precarious nature of NON-HYPERBOLIC LINEARISATIONS

Example: Polar coordinates

$\dot{r} = 0, \quad \dot{\theta} = 1$ (Non-hyperbolic centre)



HGLT ?

$\dot{x} = -y$

$\dot{y} = x$

$\alpha = 0$

$\alpha = -10^{-6}$

Small change

$\dot{r} = \alpha r$ $\qquad \alpha < 0$ changes phase portrait (no matter how small)

or

$\dot{r} = \alpha r^3$

$\alpha < 0$

non-linear change.

changes to spiral behaviour for arbitrarily small non-zero α.

Conservative.

$E(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ $\boxed{17.2}$

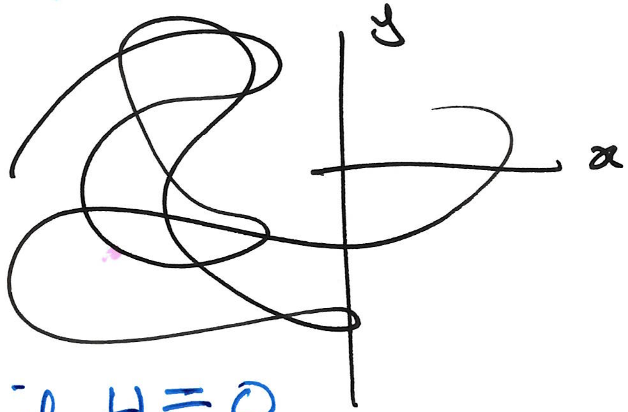
$V(x,y)$

~~Constant~~ $H(x,y)$ - constant on trajectories

$H \not\equiv \text{const}$

$H(x,y) \equiv 0 \rightarrow$ doesn't give any information

all curves are invariant if $H \equiv 0$



Recall

$\dot{x} = y, \quad \dot{y} = -x \quad , \quad H(x,y) = x^2 + y^2$, orbits were constrained to lie on circles

$$\left.\begin{array}{ll} L \to R & , y > 0 \\ R \to L & y < 0 \end{array}\right\} \dot{x} = y$$

clockwise behaviour.

Newton II $\quad m\ddot{x} = F(x)$, $m \neq 0$. $\quad \boxed{17.3}$

$$\dot{x} = y$$
$$\dot{y} = \frac{F(x)}{m}$$

$$E(x,y) = \underbrace{\tfrac{1}{2}my^2}_{KE} + \underbrace{V(x)}_{PE}$$

Def^n (5.22)

$$V(x) = \int -F(x)\,dx.$$

$$\dot{x} = \frac{dx(t)}{dt}, \quad \dot{y} = \frac{dy(t)}{dt}$$

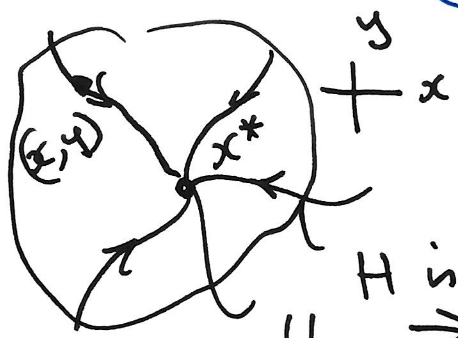
? $\dfrac{dE}{dt} = \dfrac{\partial E}{\partial x}\cdot\dot{x} + \dfrac{\partial E}{\partial y}\dot{y}$

$$= -F(x)y + my\dot{y} = -F(x)y + my\ddot{x} = y(-F(x) + m\ddot{x}) = 0$$

E is constant – energy KE + PE is preserved.

N

5.3



H(x,y) – is a constant of the motion.

H is constant on each orbit and H is continuous ⇒

H_0 at x^* ⇒ $H \equiv H_0 \quad \forall (x,y) \in N \to$ trivial constant of motion!

Note for (recall from Week 9)

$\dot{x} = y, \; \dot{y} = x - x^3$

$$\frac{dy}{dx} = \frac{x - x^3}{y} \qquad E(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = \text{constant}.$$

$$Z = E(x,y) \qquad \text{Fig 21 (a), (b)}$$

→ closed curves around min & max.

NL system is the same as the LS at the non-hyperbolic points in this because conservative

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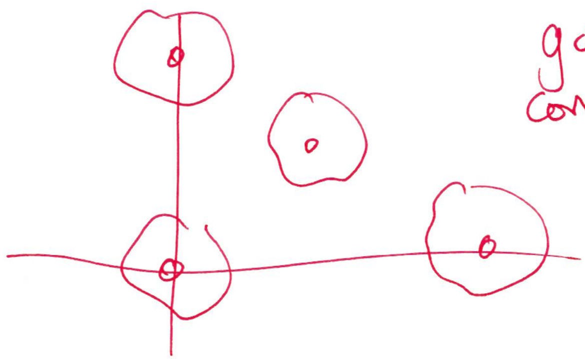
Example 5.7 → 4 points, all hyperbolic

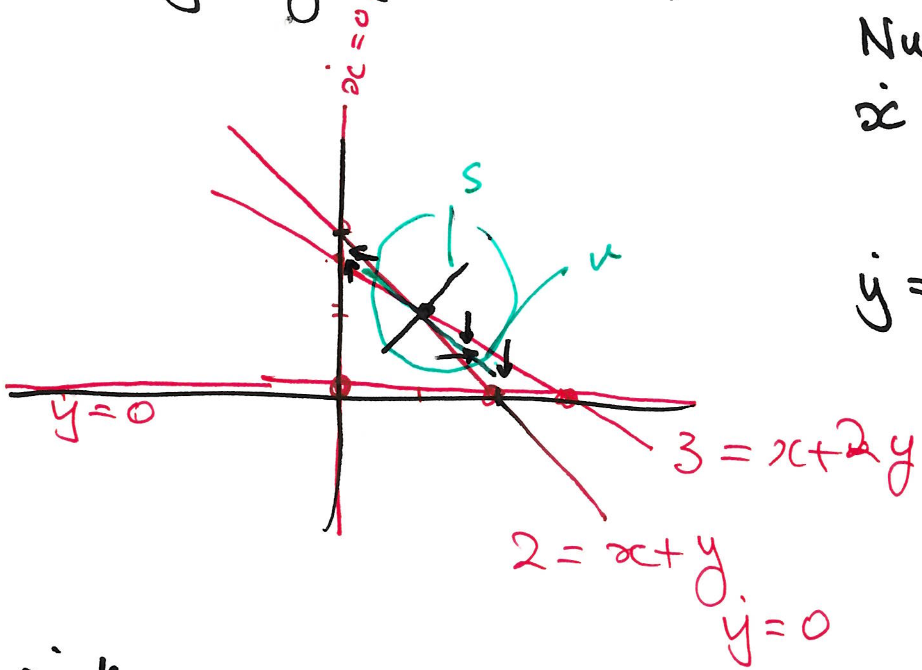


good / correct } local information at each point.

Very different HGLT give good information at each of the 4 fixed points, but how do you connect them together!

$\dot{x} = x(3-(x+2y))$ $\qquad$ $\dot{y} = y(2-(x+y))$

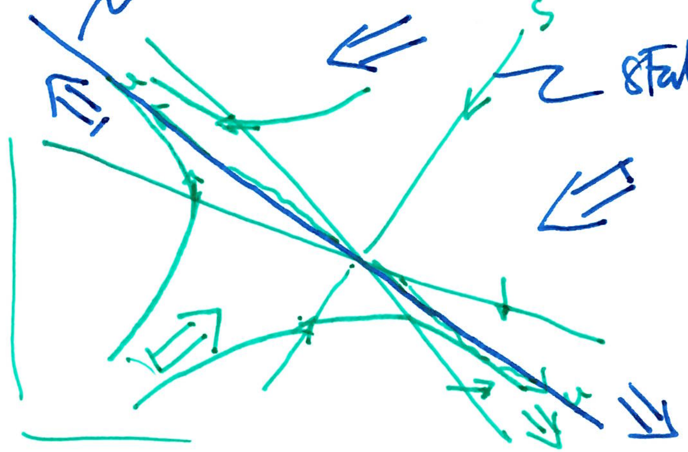
$\Rightarrow$ $\left.\begin{matrix}(0,0)\\(3,0)\\(0,2)\\(1,1)\end{matrix}\right\}$ fixed pts



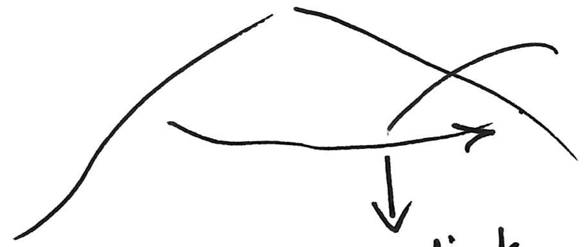
Nullcline
$\dot{x} = 0$ (vertical [illegible])

$\dot{y} = 0$ (horizontal

$(1,1)$ saddle - point



Gradient systems



$F: \mathbb{R}^2 \to \mathbb{R}$ $\quad F(x,y) \in \mathbb{R}$

$$\dot{\underline{x}} = -\nabla F(\underline{x}) = \left(-\frac{\partial F}{\partial x}, -\frac{\partial F}{\partial y}\right) \qquad \underline{x} = (x,y)$$

$$\underbrace{E(x,y) = \overset{KE}{\tfrac{1}{2}my^2} + \overset{PE}{\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2}}$$

(cf. 5.8 page 43)

energy associated with (5.23)

$$\dot{x} = -x^3 + x \quad , \dot{y} = -y$$

Gradient system for.

Gradient - lose PE as quickly as possible

Conservative keep PE + KE constant.

~~are~~ The phase portraits are $\perp$'r to each other at every point.

Thm 5.4

18.2

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$$

$$= -\dot{x}\dot{x} - \dot{y}\dot{y} \quad = -\|\dot{\underline{x}}\|^2 \qquad -|\dot{\underline{x}}|^2 \neq 0.$$

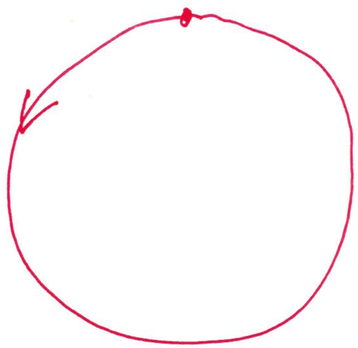
>0 for some t

$$-(\dot{x},\dot{y}) \cdot (\dot{x},\dot{y})$$

Change in F is $\Delta F = \int \frac{dF}{dt} \cdot dt < 0. \Rightarrow F(x(0)) \neq F(x(T))$ for $T \neq 0$

So no closed orbit

$\underline{x}(0) = \underline{x}(T)$



$\Rightarrow F(x(T)) = F(\underline{x}(0))$ not possible.

Think hillsides and steep slopes

$\boxed{18.3}$

Is there an F such that $\frac{\partial F}{\partial x} = f(x,y)$

$\frac{\partial F}{\partial y} = g(x,y)$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial f}{\partial y}(x,y) \qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial g}{\partial x}(x,y)$$

if $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$ the $F \nexists$

Provides a check if system is gradient to say <u>no closed orbits</u>