Lecture 15

Linear system $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$. Let $\underline{z} = \begin{pmatrix}x\\ y\end{pmatrix}$, $w = \begin{pmatrix}u\\ v\end{pmatrix}$

and suppose $\underline{z} = \underline{P}\,\underline{w}$ for some non-singular $\underline{P}$ (2×2 matrix).

So $\dot{\underline{z}} = \underline{A}\,\underline{z}$ & $\underline{P}\dot{\underline{w}} = \underline{A}\,\underline{P}\,\underline{w} \Rightarrow \dot{\underline{w}} = \underline{P}^{-1}\underline{A}\,\underline{P}\,\underline{w}$.

$\underline{P}^{-1}\underline{A}\,\underline{P}$ can be $J_1 = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$, $J_2 = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$ or $J_3 = \begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$ for suitable choice of $\underline{P}$.

Interpretation of diagram Figure 18

yellow region : $\delta < 0$ det $< 0 \Rightarrow \lambda_1, \lambda_2$ real and opposite sign (SADDLE)

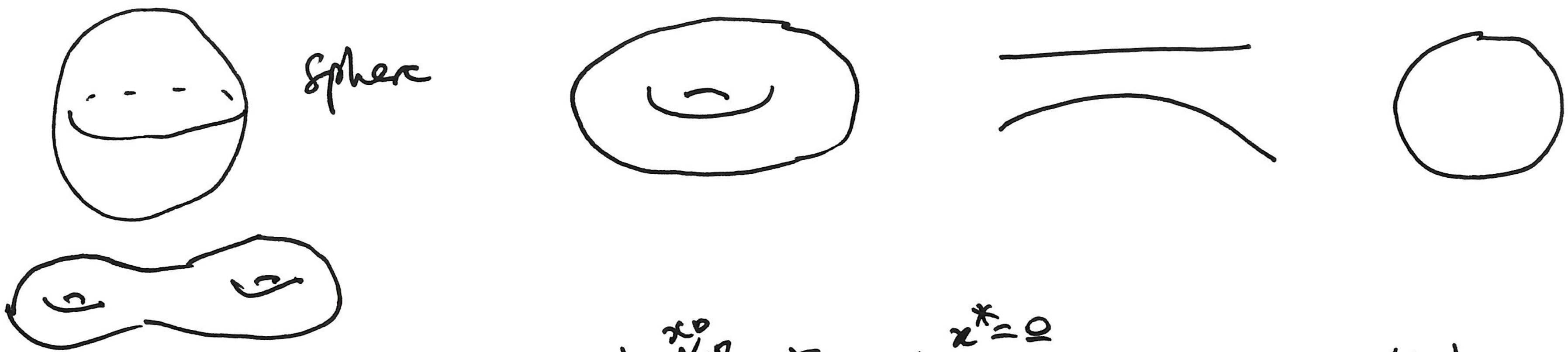
orange/green region : $\tau^2 < 4\delta \Rightarrow \lambda_1, \lambda_2$ complex $\lambda_{1,2} = \alpha + i\beta$, $\beta \neq 0$.
$\alpha < 0$ stable spiral, $\alpha > 0$ unstable spiral

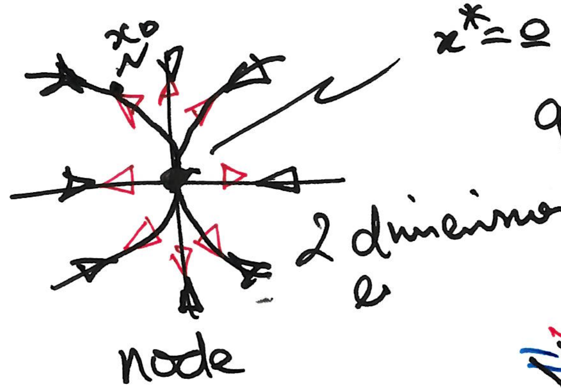
red region : $\tau^2 > 4\delta$, $\tau > 0$, λ_1, λ_2 real and positive (UNSTABLE NODE)

blue region : $\tau^2 < 4\delta$, $\tau < 0$, λ_1, λ_2 real and negative (STABLE NODE)

Curve $\tau^2 = 4\delta$: $\lambda_1 = \lambda_2 (= \lambda)$ if A is diagonal $J_1 = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$ ($= A$ actually!)
A is not diagonal $J_2 = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$

Manifolds a top space which is locally Euclidean (real $\boxed{15.2}$
n-dimensional space (locally $\mathbb{R}^n$, for some n)





9 soln curves/dot.

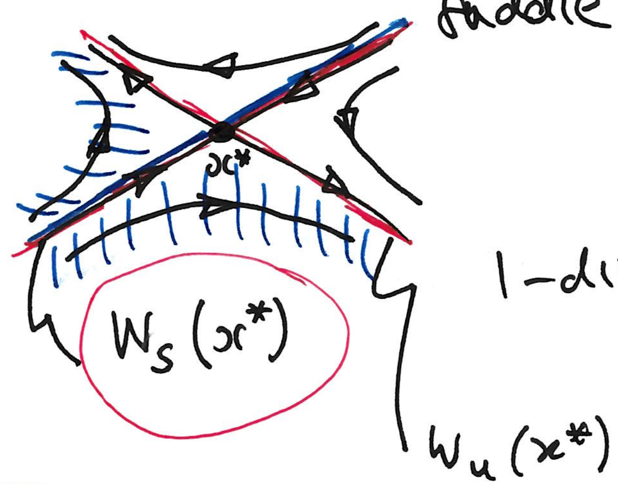
2 dimensional

Stable manifold

Unstable manifold.

$W_s(x^*) = \mathbb{R}^2$

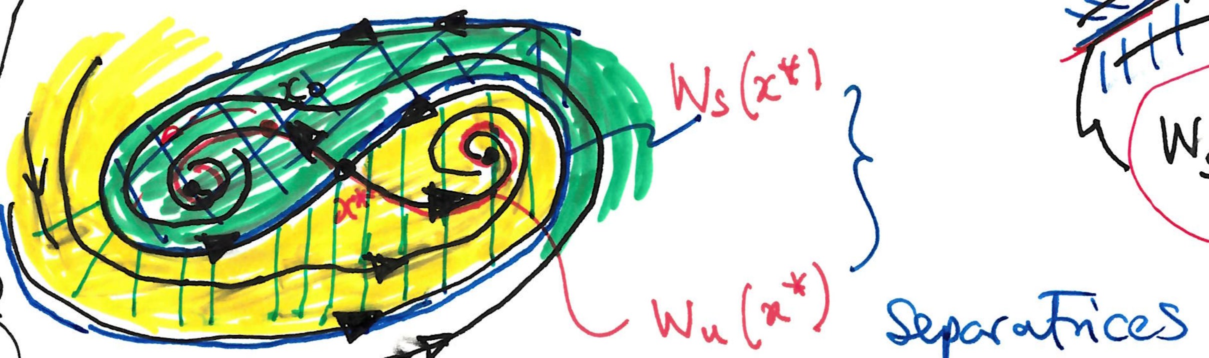
$W_u(x^*) = \{\underline{0}\}$



1-dimensional.

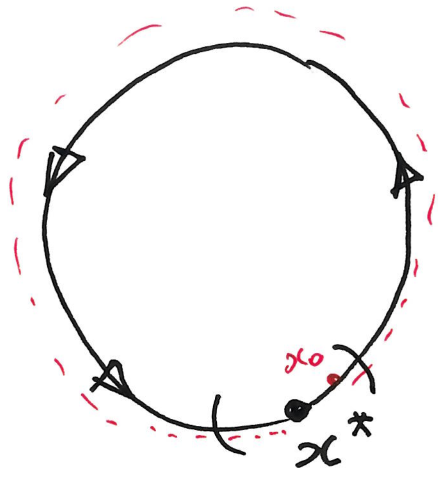
basin of right hand fixed pt

basin of left hand fixedpoint



The two basins are SEPARATED by the stable manifold of the saddle.

<u>Attracting point</u> x^* has a nbd N such that all $x(t)$ with $x_0 \in N$ ~~satisf~~ satisfy $x(t) \to x^*$ as $t \to \infty$.

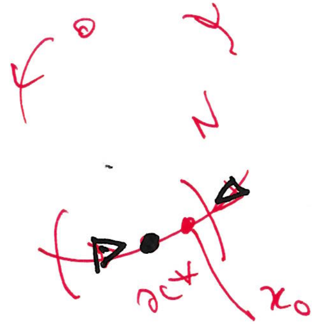


not AS ×

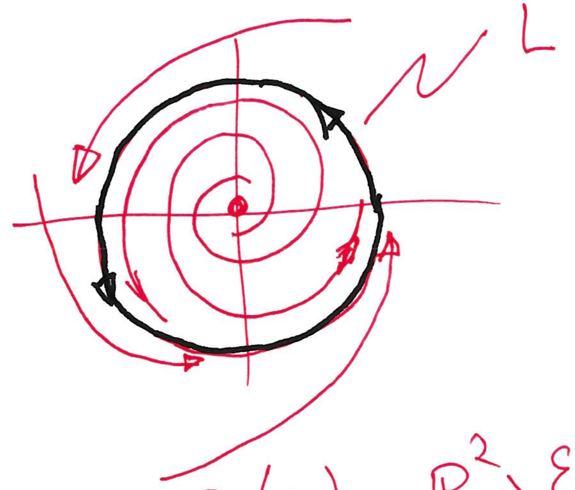
<u>attracting</u> ✓

soln curve on S^1

<u>$x(t)$</u> $x(0) = x_0$



AS ✓



$B(L) = \mathbb{R}^2 \setminus \{0\}$

16.1

Lecture 16 (Chapter 5)

1,2,3 $\dot{x} = f(x)$, $x \in \mathbb{R}$, $x \in S$

4 $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$, $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $(x,y) \in \mathbb{R}^2$.

5 $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix}$, $f,g : \mathbb{R}^2 \to \mathbb{R}$ $(x,y) \in \mathbb{R}^2$.

FPs $\underset{\mathbb{R}^2}{(x,y)} = (x^*, y^*)$ is a fixed point if $f(x^*,y^*) = 0$ $(\updownarrow)$

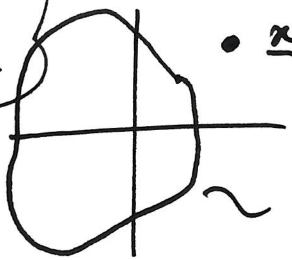
$g(x^*,y^*) = 0$ $(\leftrightarrow)$

fixed pt.

Def^n 5.1

5.2

- $\underline{x}^* = (x^*, y^*)$



~ periodic orbit of period T

$\underline{x}(t)$ is a solution of $\dot{\underline{x}} = (f(\underline{x}), g(\underline{x}))$ with

$\underline{x}(t) \equiv \underline{x}(t+T)$, $T > 0$, minimum positive T.

If we allowed $T = 0$, no constraint.

In one dim S

$\dot{\theta} = 1$, periodic orbit

(of period 2π rads)

<u>Existence & Uniqueness</u> $\dot{\underline{x}} = \underline{f}(\underline{x})$ * $\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2$ $\boxed{16.2}$

$\underline{f}$ is cont. (in U open set) given $x_0 \in U$, $\exists$ $x(t)$ s.t. $x(0) = x_0$ and $x(t)$ exists for some open interval containing $0 \in \mathbb{R}$

$\underline{f}$ is differentiable $x(t)$ is unique

$x(t)$, $x'(t)$ are soln of *

with $\underline{x}(0) = \underline{x}_0 = x'(0)$

then $x(t) \equiv x'(t)$ on an interval containing $t = 0$.

(true for n-dim).

<u>Technique 1</u> <u>Linearisation</u> (linear stability on $\mathbb{R}$!? ☺)

$f(x_0) = 0$, for $\dot{x} = f(x)$,

$f'(x_0) < 0$, linear stab

$f'(x_0) > 0$, linear unstable

$f'(x_0) = 0$ Not sure!

5.2 Linearisation in $\mathbb{R}^2$. (FP at $(x,y)=(x^*,y^*)$ $(=\underline{x}^*)$) $\boxed{16.3}$

$$\dot{x} = f(x,y) = f(x^*,y^*) + \frac{\partial f}{\partial x}(\underline{x}^*)(x-x^*) + \frac{\partial f}{\partial y}(\underline{x}^*)(y-y^*) + O(3)$$

$$\dot{y} = g(x,y) = g(x^*,y^*) + \frac{\partial g}{\partial x}(\underline{x}^*)(x-x^*) + \frac{\partial g}{\partial y}(\underline{x}^*)(y-y^*) + O(3)$$

$$x-x^*=u : \dot{x} = \dot{u} = 0 + \frac{\partial f}{\partial x}(\underline{x}^*)\,u + \frac{\partial f}{\partial y}(\underline{x}^*)\,v + \cancel{h.o.t.}$$

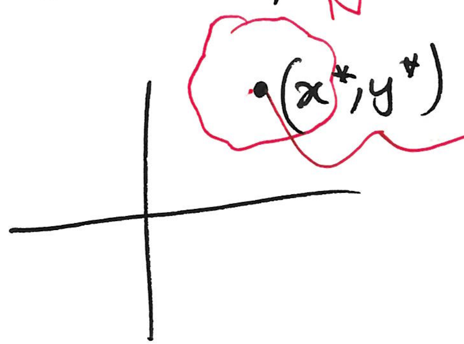
$$y-y^*=v : \dot{y} = \dot{v} = 0 + \frac{\partial g}{\partial x}(\underline{x}^*)\,u + \frac{\partial g}{\partial y}(\underline{x}^*)\,v + \cancel{h.o.t}$$

$\downarrow$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \underbrace{\frac{\partial(f,g)}{\partial(u,v)} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}_{\underline{x}^*}}_{\text{Jacobian matrix}} \begin{bmatrix}u\\ v\end{bmatrix}$$

"LINEARISED SYSTEM"

$\rightarrow$ saddle, node, centre, spiral, line of fixed pts.



linear system ? useful information

Hartman–Grobman Theorem

Hyperbolic fixed points: if none of the eigenvalues of A have a zero real part.

J_1: $\lambda_1, \lambda_2 \neq 0 \leftarrow A$; "wobble" A and it will remain in the same J_i category.

$\mathcal{E}(A) = \mathcal{E}(J)$
$J = P^{-1}AP$ (Ch 4)

J_2 $\quad \lambda \neq 0$

J_3 $\quad \lambda_1, \lambda_2 = \alpha \pm i\beta, \quad \alpha \neq 0$

These conditions are maintained for SUFFICIENTLY small changes in the matrix – the matrix can be said to "structurally stable" – no change of type of Jordan form for sufficiently small perturbation

H-G Lin. Theorem (5.2) p36

[If J is hyperbolic ($\therefore$ (A) also), then the NL and Linearised systems are the same qualitatively on a sufficiently small nbd of the fixed point (5.2)]

[16.5]

So HGLT is "worrying" if $\exists\, \lambda(DJ(\underline{x}))$ is not hyperbolic – the theorem gives no conclusion.

$$\dot{x} = y - x^3 \;, \quad \dot{y} = -x - y^3$$

FPs : $y = x^3$; $x = -y^3 \Rightarrow y = -y^9 \Rightarrow y = 0, x = 0$

$\therefore$ Single fixed point $(x,y) = (0,0)$

$$\text{Jacobian } J = Df(\underline{x})\Big|_{\underline{x}=0} = \begin{bmatrix} -3x^2 & 1 \\ -1 & -3y^2 \end{bmatrix}\Bigg|_{\underline{x}=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$\mathcal{E}(Df(\underline{0}))$: $\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$, $\beta = 1$, $\alpha = 0$ (nonhyperbolic)

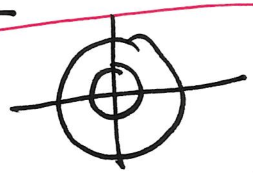
$\longrightarrow$ centre (ex of nhyper lin. sys)

$$\dot{r} = \alpha r$$
$$\dot{\theta} = \beta$$

HGLT doesn't apply

Change to polar coords $\quad r\dot{r} = xy - x^4 + y(-x - y^3)$
$= -x^4 - y^4$ $\quad \dot{r} = -\frac{(x^4 + y^4)}{r} < 0$ for $r \neq 0$

Origin is a stable fixed pt (spiral?)

HGLT 

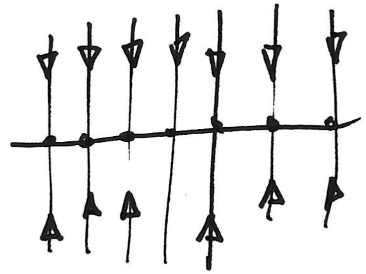
reality  $\dot{r} < 0$ stability

Ex 5.2 $\qquad \boxed{16.6}$

$\dot{x} = -x^2, \quad \dot{y} = -y$ $\qquad$ FPs $\quad x = y = 0 \Rightarrow (x, y) = (0, 0)$

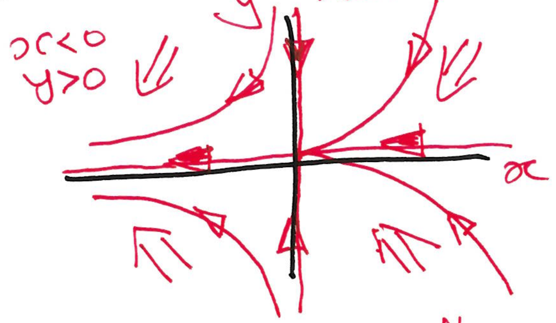
$\underline{f}(\underline{x}) = (-x^2, -y)$ $\qquad$ $D\underline{f}(\underline{x}) = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

<u>Linear system</u>



non-hyperbolic

NL system



$x = 0, \ \dot{x} = 0 \ \updownarrow$

$y = 0, \ \dot{y} = 0 \ \leftrightarrow$

reality

saddle-node