Lecture 13 Linear system in the plane. $\boxed{13.1}$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (x,y)\in\mathbb{R}^2 \qquad \text{“}[\;] = (\;)\text{”}$$

$$\underline{z} = \underline{P}\,\underline{w} \qquad P^{-1}\ \exists \quad \underline{w} = P^{-1}\underline{z} \quad \text{1-1 bijection,}$$

$$z = \begin{bmatrix}x\\ y\end{bmatrix} \quad \underline{w} = \begin{bmatrix}u\\ v\end{bmatrix}$$

non-singular reversible cf. $x \mapsto x^2$

$-3 \longmapsto 9$

$\left\{\begin{matrix}3\\ -3\end{matrix}\right. \longleftarrow 9$ which one?

$$\dot{\underline{z}} = \underline{A}\,\underline{z} \iff \dot{\underline{w}} = \underline{P}^{-1}\underline{A}\,P\,\underline{w}$$

$$\| $$

$$P\dot{\underline{w}} = \underline{A}\,\underline{P}\,\underline{w} \qquad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \to \underline{P}^{-1}\underline{A}\,\underline{P}\ ?$$

<u>Jordan</u>: choose $\underline{P}$ s.t. $\underline{P}^{-1}\underline{A}\,\underline{P} = J_1, J_2, J_3$

$$J_1 = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}, \quad J_2 = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}, \quad J_3 = \begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$$

~~Congu~~ Conjugate matrices have the same eigenvalues

J_1 eigenvalues λ_1, λ_2 — $\lambda_1 \neq \lambda_2$ - diagonal; $\lambda_1 = \lambda_2$ could be diagonal J_1 or J_2

J_2 — λ, λ

J_3 — $\alpha \pm i\beta$ — $J_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $\quad \begin{vmatrix} \lambda-\alpha & \beta \\ -\beta & \lambda-\alpha \end{vmatrix} = 0$

$(\lambda-\alpha)^2 + \beta^2 = 0$

$\lambda = \alpha \pm i\beta$

Choose P to get J_1, J_2, J_3.

$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\quad \lambda_1 \neq \lambda_2$ Evals $\quad \underline{v}_1, \underline{v}_2$ Evecs

Eigenvector eq^n is $\quad A\underline{v}_1 = \lambda_1 \underline{v}_1$

$A\underline{v}_2 = \lambda_2 \underline{v}_2$

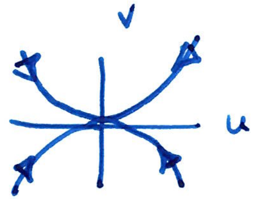
Let

$\underline{P} = [\underline{v}_1 \,\vdots\, \underline{v}_2]$ $\quad AP = A[\underline{v}_1 \,\vdots\, \underline{v}_2] = [A\underline{v}_1 \,\vdots\, A\underline{v}_2]$

$= [\lambda_1\underline{v}_1 \,\vdots\, \lambda_2\underline{v}_2] = [\underline{v}_1 \,\vdots\, \underline{v}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P\underline{J}_1$

Ex to check "=" $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \underline{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\quad \lambda_1 = 9$, $\lambda_2 = 7$ $\quad \Rightarrow \quad \underline{P}^{-1}\underline{A}\underline{P} = J_1$

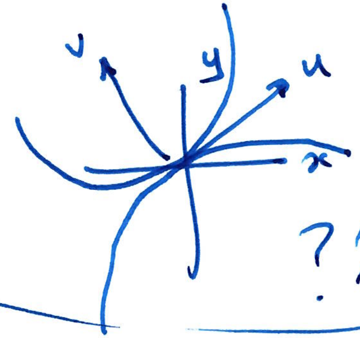
$$\underline{\dot{w}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \underline{J_1}\, \underline{w}$$



$$\frac{dv}{du} = \frac{\lambda_2}{\lambda_1} \frac{v}{u} \rightarrow$$ nodes, saddles

stable node ?
unstable
saddle $\lambda_1 \lambda_2 < 0$

How is $z = \begin{bmatrix} x \\ y \end{bmatrix}$ related to $\begin{bmatrix} u \\ v \end{bmatrix}$?



??

$$J_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$ Evals λ, repeated

$A\underline{v}_1 = \lambda \underline{v}_1$ - just one. $\underline{v}_1$ = eigenvector.

Second vector $\underline{v}_2$ $(A - \lambda I)\underline{v}_2 = \underline{v}_1$ * find $\underline{v}_2$ after $\underline{v}_1$

$\underline{P} = [\underline{v}_1 \vdots \underline{v}_2]$, $A\underline{P} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} [\underline{v}_1 \vdots \underline{v}_2] = [\lambda \underline{v}_1 \vdots \underline{v}_1 + \lambda \underline{v}_2]$

$$= [\underline{v}_1 \,\vdots\, \underline{v}_2] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \underline{P}\,\underline{J}_2 \qquad \underline{P}^{-1}\underline{A}\,\underline{P} = J_2$$

$\underline{J_3}$

$\lambda_1 = \alpha - i\beta$, $\lambda_2 = \alpha + i\beta$, $\beta \neq 0$

$$\underline{P}^{-1}\underline{A}\,\underline{P} = J_3$$

$$J_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$\downarrow$ eigenvector $\qquad \downarrow$ eigenvector

eigenvector (λ_1)

$\underline{v}_1 + i\underline{v}_2$ $\longrightarrow$ $A(\underline{v}_1 + i\underline{v}_2) = (\alpha - i\beta)(\underline{v}_1 + i\underline{v}_2)$

real and imaginary parts

$$\left.\begin{aligned} \underline{A}\underline{v}_1 &= \alpha \underline{v}_1 + \beta \underline{v}_2 \\ A\underline{v}_2 &= -\beta \underline{v}_1 + \alpha \underline{v}_2 \end{aligned}\right\}$$

taking real & imag parts of eigval-eigvec eqn.

$$\underline{P} = [v_1 \,\vdots\, v_2] \Rightarrow$$

$$AP = A[\underline{v}_1 \,\vdots\, v_2] = [A\underline{v}_1 \,\vdots\, A\underline{v}_2] = [\alpha \underline{v}_1 + \beta v_2 \,\vdots\, -\beta \underline{v}_1 + \alpha \underline{v}_2]$$

$$= [v_1 \,\vdots\, v_2]\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \underline{P}\,\underline{J}_3$$

$$\underline{P}^{-1}\underline{A}\,\underline{P} = \underline{J}_3$$

Note: eigenvectors are not unique but give a unique direction. $\boxed{14.1}$

$A\underline{v}_1 = \lambda_1 \underline{v}_1$ $v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ or $v_1' = \begin{bmatrix}2\\4\end{bmatrix}$ $v_1' = 2\begin{bmatrix}2\\4\end{bmatrix}$

↓

eigenvector

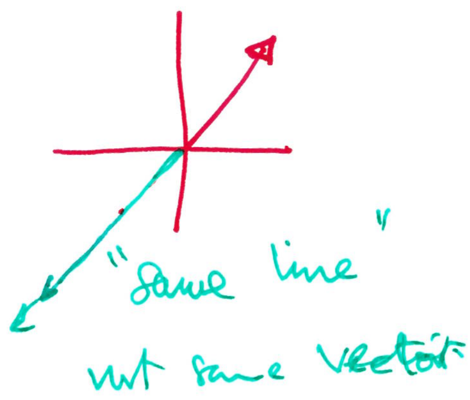
↓

eigenline / eigendirection

$2A\underline{v}_1 = 2\lambda_1 \underline{v}_1$

$A(2\underline{v}_1) = \lambda_1(2v_1)$

"Choose (non-zero)"



LECTURE 14

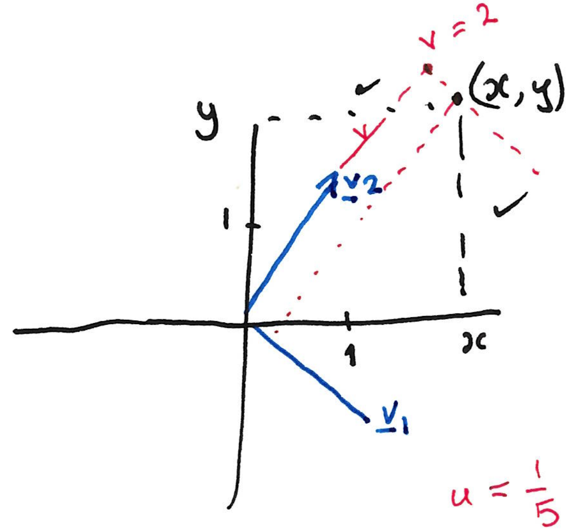
$\underline{z} = \begin{bmatrix}x\\y\end{bmatrix}$ ✓ $\underline{w} = \begin{bmatrix}u\\v\end{bmatrix}$ ✓ $\underline{z} = \underline{P}\underline{w} \longrightarrow P = [\underline{v}_1 \,\vdots\, \underline{v}_2]$

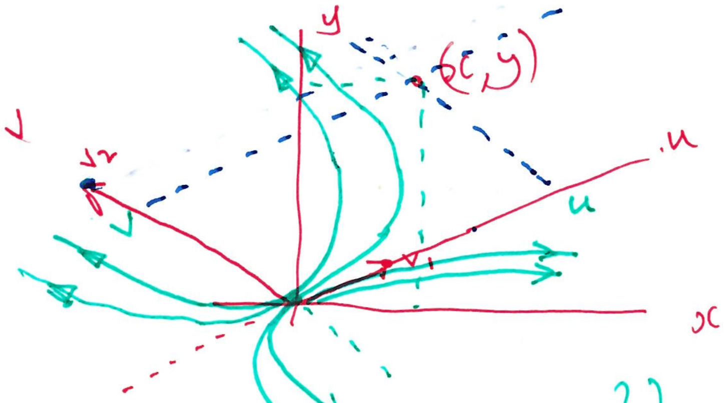
$x = u_1$

$\begin{bmatrix}x\\y\end{bmatrix} = [\underline{v}_1 \,\vdots\, v_2]\begin{bmatrix}u\\v\end{bmatrix}$

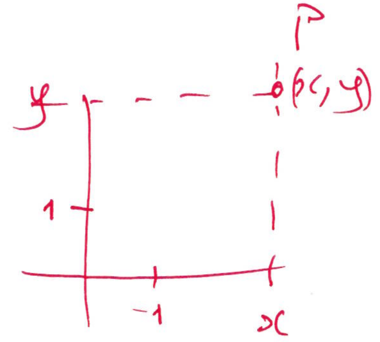
$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = u\begin{bmatrix}\underline{v}_1\end{bmatrix} + v\begin{bmatrix}\underline{v}_2\end{bmatrix}$

$\underline{z} = \underline{P}\,\underline{w}$





Change of coordinates relative to new basis.

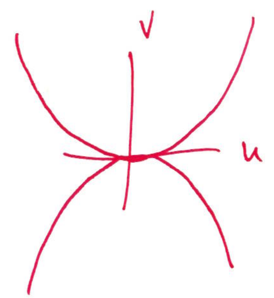


$(x, y) = 0.8\underline{v}_2 + 3\underline{v}_1$?? approx.
$v \qquad u$

$$\dot{u} = u$$
$$\dot{v} = 2v$$

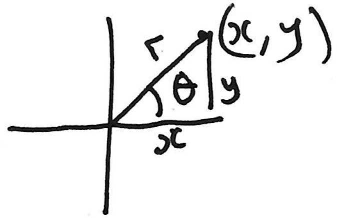
$$\frac{dv}{du} = \frac{2v}{u}$$

$$\ln v = 2\ln u + C$$
$$\underline{v = Bu^2}$$



✓

Polar coordinates - non-linear



$(r, \theta) \longleftrightarrow (x, y)$

$$\left.\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{array}\right\}$$ dynamical implications of these relations

14.3

Diff w.r.t. time

$$r^2 = x^2 + y^2 \rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\boxed{r\dot{r} = x\dot{x} + y\dot{y}}$$

$$\tan\theta = y/x \rightarrow \sec^2\theta\,\dot{\theta} = \frac{d}{dt}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

$$\Rightarrow \dot{\theta} = \cos^2\theta\,\frac{x\dot{y} - y\dot{x}}{x^2}$$

$x = r\cos\theta$

$$= \frac{x\dot{y} - y\dot{x}}{r^2}$$

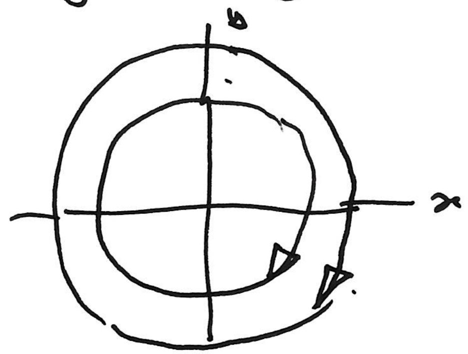
$$\boxed{r^2\dot{\theta} = x\dot{y} - y\dot{x}}$$

$\dot{x} = y,\ \dot{y} = -x$: $r^2\dot{\theta} = -x^2 - y^2$ $\qquad \dot{\theta} = -1$

$$r\dot{r} = x\dot{x} + y\dot{y} = xy + -xy = 0$$

$\dot{r} = 0$

$\dot{x} = y,\ \dot{y} = -x \rightarrow \dot{\theta} = -1,\ \dot{r} = 0$



114.4 NLinear

$$\dot{x} = y - x(x^2+y^2) \qquad \dot{y} = -x - y(x^2+y^2)$$

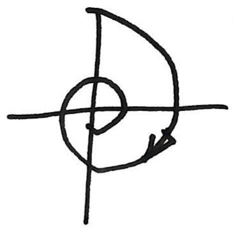
$$r\dot{r} = x\dot{x} + y\dot{y} = \cancel{xy} - x^2r^2 - \cancel{xy} - y^2(r^2)$$

$$r\dot{r} = -r^4, \quad \dot{r} = -r^3$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = -x^2 - \cancel{xyr^2} - y^2 + \cancel{xyr^2}$$

$$= -r^2$$

stable spiral



$$\dot{r} = -r^3, \dot{\theta} = -1$$

Linear example

$$\left.\begin{aligned} \dot{x} &= \alpha x - \beta y \\ \dot{y} &= \beta x + \alpha y \end{aligned}\right\} \quad J_3 \text{ system}$$

$$'' \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\dot{x} = -x^3$$

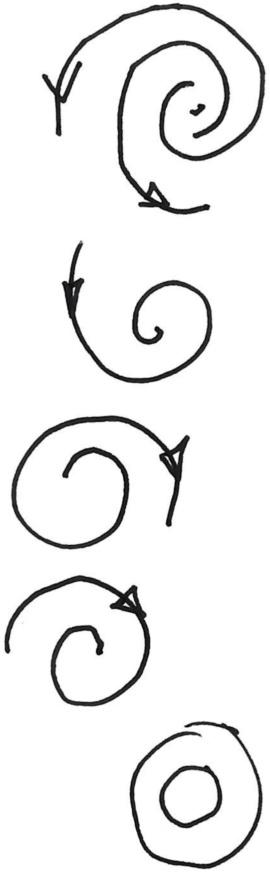
Ex

$$\dot{r} = \alpha r$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{\cancel{\alpha}\,\beta x^2 + \alpha yx - \alpha yx + \beta y^2}{r^2} = \frac{\beta(x^2+y^2)}{r^2} = \beta$$

$\dot{r} = \alpha r$

$\dot{\theta} = \beta$



$\alpha > 0, \beta > 0$	unstable spiral	(AC)
		(AS)
$\alpha < 0, \beta > 0$	stable spiral	(AC)
$\alpha > 0, \beta < 0$	unstable spiral	(C) (AS)
$\alpha < 0 \quad \beta < 0$	stable spiral	(C)
$\alpha = 0$	centre	(stable, not AS)

4.4 p31 Tr(A) – Det(A) classification of linear systems

Eigenvalues of $\underline{A}$ $\quad \mathrm{Det}(\lambda I - A) = 0 \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0 \qquad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \mathrm{Det}(A) = 0$$

14.6.

$$\lambda = \frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4\,Det(A)}}{2}$$ – Eigenvalues of A.

Ex 4.4 p32 – work through.