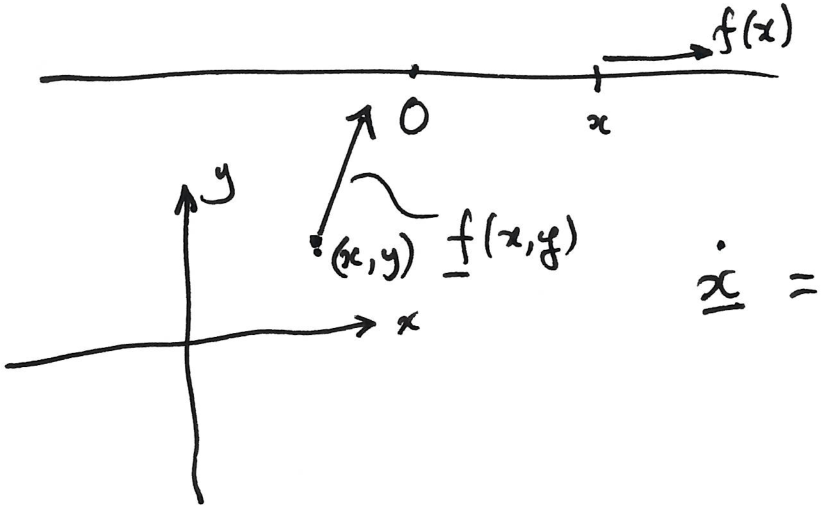
Lecture 11 - Linear systems in the plane $\mathbb{R}^2$.



$\mathbb{R}$ $\qquad \dot{x} = f(x)$

$\dot{\underline{x}} = \underline{f}(\underline{x})$ $\qquad \underline{x} \in \mathbb{R}^2$

$\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2.$

$\underline{f}(\underline{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\underline{x} = (1, 1)$

$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ X, $\quad \underline{z} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\dot{\underline{z}} = A \underline{z} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Eigenvalues of $\underline{A}$ $\quad \det(\lambda \underline{I} - \underline{A}) = 0$, λ_1, λ_2 two eigenvalues

Eigenvectors $\underline{w}$ $\quad \underline{A}\underline{w}_1 = \lambda_1 \underline{w}_1$, $\underline{A}\underline{w}_2 = \lambda_2 \underline{w}_2$

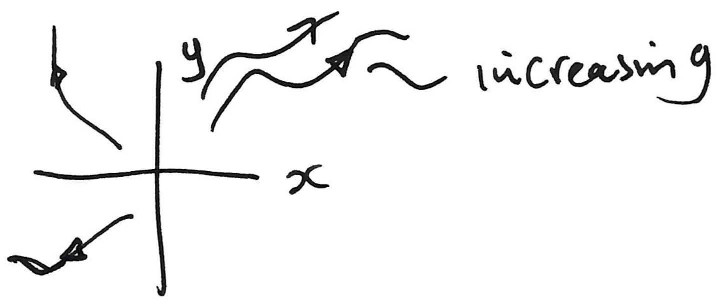
$(\underline{w}_1, \underline{w}_2)$ may be linearly dep. (scalar mult of $\underline{w}_1/\underline{w}_2 = \underline{w}_2/\underline{w}_1$

$$\det(\lambda I\ A) = \lambda^2 - \mathrm{Tr}(A)\lambda + \mathrm{Det}(A) = 0. \quad (4.4)$$

Ex 4.1 (+)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

" [= ("



$$\frac{dx}{dt} = \lambda_1 x \qquad x = x(t)$$

$$\frac{dy}{dt} = \lambda_2 y \qquad y = y(t)$$

eliminate t ?

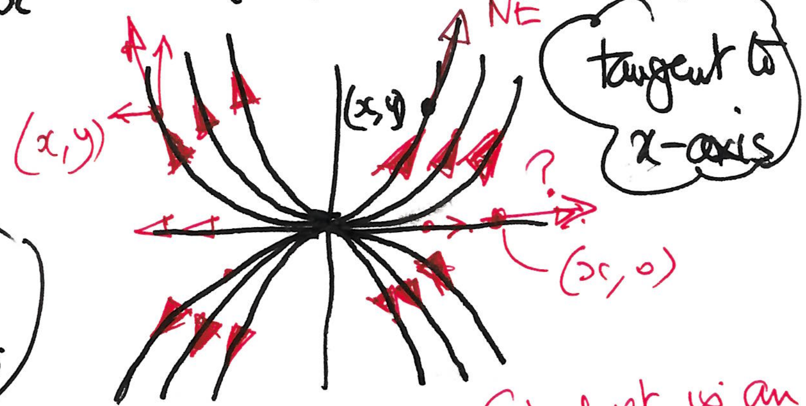
11. 3.

$$\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\lambda_2 y}{\lambda_1 x} \Rightarrow \frac{dy}{dx} = \frac{\lambda_2 y}{\lambda_1 x} \Rightarrow \int \frac{dy}{y} = \int \frac{\lambda_2}{\lambda_1} \frac{dx}{x}$$

$$\ln y = \frac{\lambda_2}{\lambda_1} \ln x + C \qquad C = \ln(B)$$

$$y = B x^{\lambda_2/\lambda_1}$$

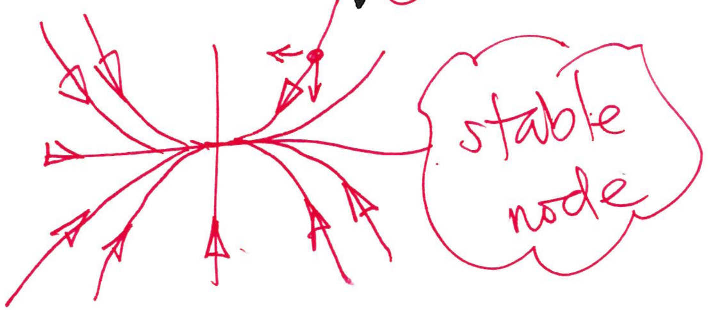
A $\dfrac{\lambda_2}{\lambda_1} > 0$ $\quad k = \dfrac{\lambda_2}{\lambda_1} > 1 \quad y = B x^{k}$ (slope 0 at $x = 0$)



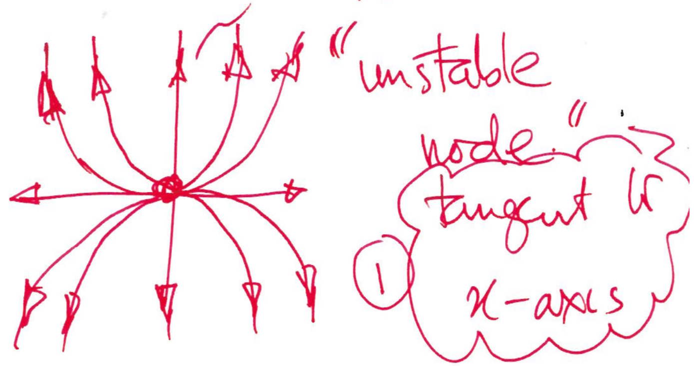
$\dot{x} = \lambda_1 x$

$\dot{y} = \lambda_2 \cancel{x}$

$\lambda_1 > 0$, $\lambda_2 > 0$ ① or ② $\lambda_1 < 0$, $\lambda_2 < 0$.



fixed pt is an "unstable node"



11. 4

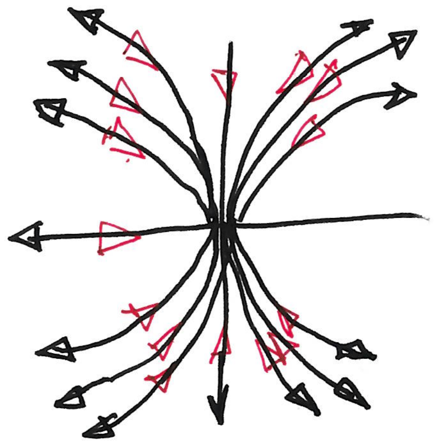
$0 < \frac{\lambda_2}{\lambda_1} < 1$

$\frac{\lambda_2}{\lambda_1} = k$

$y = Bx^k$, $0 < k < 1$

$y' = Bkx^{k-1}$ ($\to \infty$ at $x = 0$)

$\frac{1}{B^{1/k}} y^{1/k} = x$ $\quad 1/k > 1$



$\lambda_1, \lambda_2 > 0$ unstable node (tangent to y axis)

$\lambda_1, \lambda_2 < 0$ stable node (tangent to $\cancel{x}$ y-axis)

B

$\frac{\lambda_2}{\lambda_1} < 0$

$y = Bx^k$, $k < 0$.

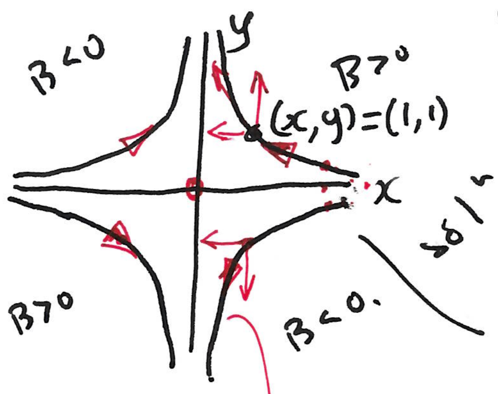
$k = -1$

$y = \frac{B}{x}$

11. 5

$y = \frac{B}{x}$

$B = 1$



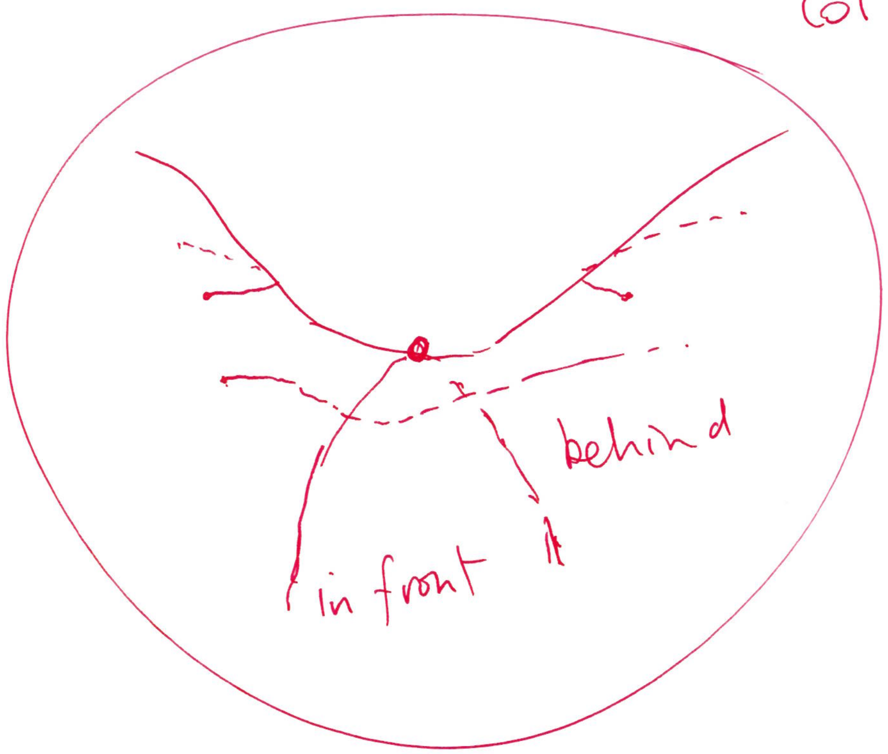
Suppose $k = -1$ possibly $\left.\begin{array}{l}\lambda_1 = -1 \\ \lambda_2 = +1\end{array}\right\}$

$\dot{x} = \cancel{xy} - x$

$\dot{y} = \cancel{x}\ y$

"saddle" fixed

level contours of a landscape at a "col"



Lecture 12

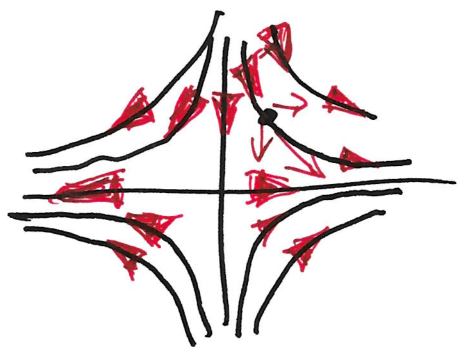
Midterm Wednesday 9th November, Test access 10.00 to 13.05 Once submitted, no resubmission

Submit, in addition to the quiz, your workings in support of the quiz. (Has to be submitted by 13.05 also). (Voluntary)

10 questions 10 marks each.

$\dot{x} = \lambda_1 x$, $\dot{y} = \lambda_2 y$

I



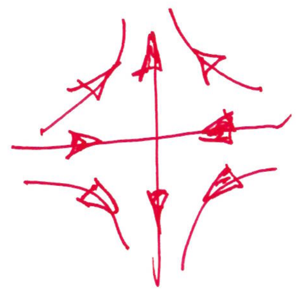
$\frac{\lambda_2}{\lambda_1} < 0$

I

$\lambda_1 > 0$
$\lambda_2 < 0$

$\lambda_1 < 0$
$\lambda_2 > 0$

Let $\frac{\lambda_2}{\lambda_1} = -1$ for an example. $y = Bx^k$ $k < 0$.

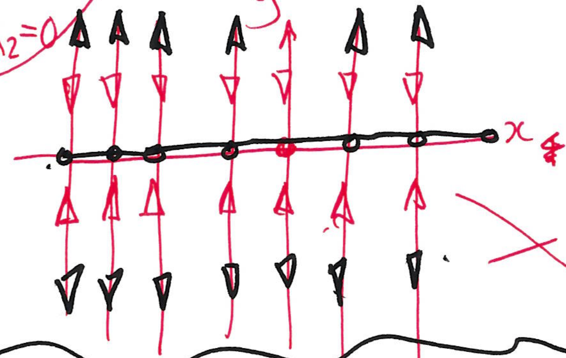


12.2

$\lambda_1 = 0$, $\lambda_2 \neq 0$ $\Rightarrow$ obvious corresponding conclusions for $\lambda_1 \neq 0$, $\lambda_2 = 0$

$\dot{x} = 0$, $\dot{y} = \ldots$

$\dot{x} = 0$, $\dot{y} = 0$ $\Rightarrow$ $y = 0$



$\lambda_2 > 0$

$\lambda_2 < 0$

Warning – two phase portraits on this diagram

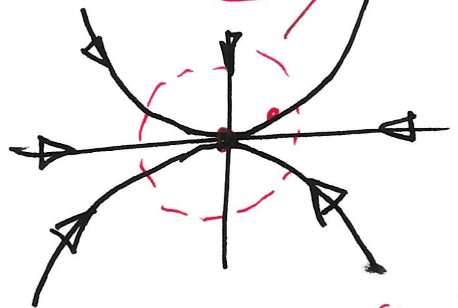
$\lambda_2 > 0$

$\lambda_1 \neq 0$ $\lambda_2 = 0$

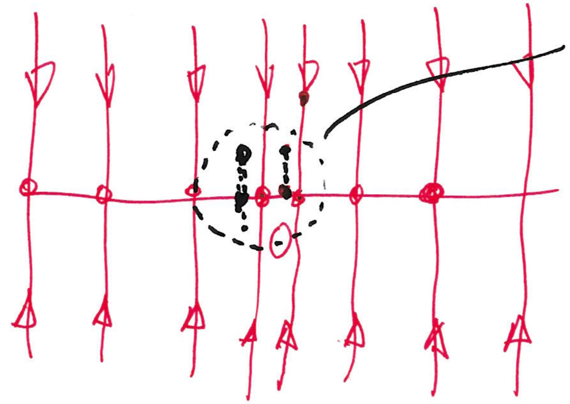
stable node

it asymptotically stable

N



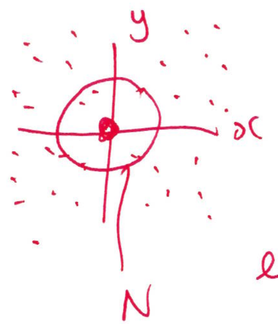
as t increase, orbit stay in N asymptotically approaches the origin



$\lambda_2 < 0$
$\lambda_1 = 0$

stability / $\underline{\underline{\text{not}}}$ asymptotic stability

$\lambda_1 = 0 \;, \lambda_2 = 0$



$\dot{x} = \lambda_1 x = 0$

$\dot{y} = \lambda_2 y = 0$

every pair is a fixed point incl $\underline{0}$

do orbits stay in N

yes, they don't move!!

Is $\underline{0}$ AS No – no asymptotic movement of the nbd of pts towards $\underline{0}$.

AS = asymptotically stable

T2.3

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p26

$$\boxed{\dot{\underline{z}} = \underline{A}\,\underline{z}} \qquad \underline{z} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\underline{w} = \begin{pmatrix} u \\ v \end{pmatrix}$

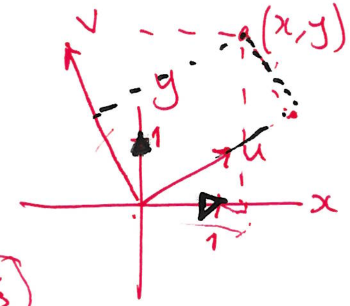
Let $\underline{z} = \underline{P}\,\underline{w}$, $P^{-1}\,\exists$

old coordinates

new coordinates

(1.5, 2) (2, ½)

$(x, y) \,(= (u, v))$



$\dot{\underline{z}} = \underline{P}\,\dot{\underline{w}} = \underline{A}\,\underline{z}$

$= A\,P\,\underline{w}$

$\underline{P}\,\dot{\underline{w}} = \underline{A}\,\underline{P}\,\underline{w} \Rightarrow \dot{\underline{w}} = \underline{P}^{-1}\underline{A}\,\underline{P}\,\underline{w}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

How can we choose $\underline{P}$ to make $\underline{P}^{-1}\underline{A}\underline{P}$ "nicer" than $\underline{A}$? 12.4

Jordan normal form is the answer —

Given $\underline{A}$, we can choose a non-singular matrix $\underline{P}$ such $\underbrace{\underline{P}^{-1}\underline{A}\underline{P}} = \underline{J}$, "Jordan" matrix

<u>Eigenvalues of A.</u>

$$J_1 = \underline{P}^{-1}\underline{A}\underline{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \underline{\lambda_1, \lambda_2 \text{ real}} \qquad \mathcal{E}(A) = \{\lambda_1, \lambda_2\}$$

$$J_2 = \underline{P}^{-1}\underline{A}\underline{P} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \underline{\lambda \text{ real}} \qquad \mathcal{E}(A) = \{\lambda, \lambda\}$$

$$J_3 = \underline{P}^{-1}\underline{A}\underline{P} = \begin{pmatrix} \alpha & +\beta \\ -\beta & \alpha \end{pmatrix} \qquad \alpha, \beta \text{ real.} \qquad \mathcal{E}(A) = \{\underset{\lambda_1}{\alpha + i\beta}, \underset{\lambda_2}{\alpha - i\beta}\}$$

<u>Thm</u> "Eigenvalues of similar matrices are the same."

see