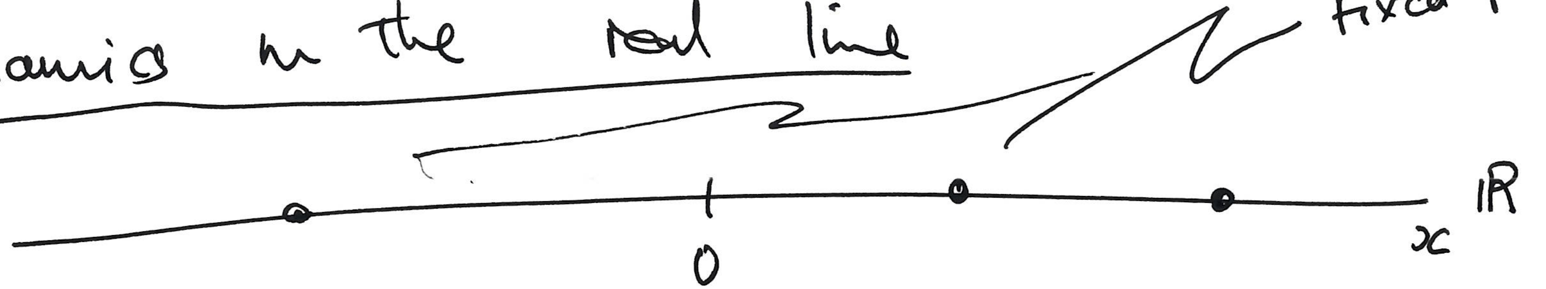
Lecture 9 - Dynamics on the circle

Dynamics on the real line



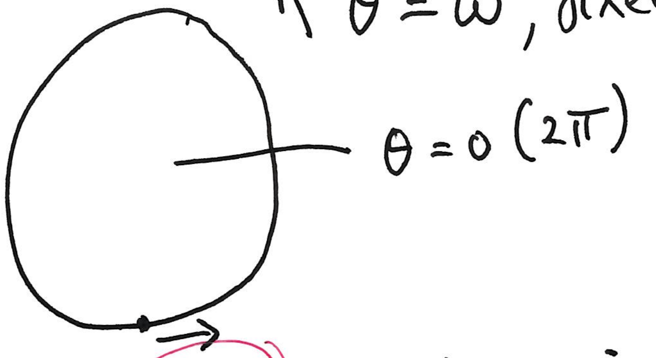
$\frac{dx}{dt} = 0$ at FPs

$\frac{dx}{dt} > 0$ Right $\rightarrow$

$\frac{dx}{dt} < 0$ Left $\leftarrow$

Dynamics on the circle

$\dot{\theta} = \omega$, fixed - uniform rotation, constant angular speed. ω



linear speed metres/sec

angular speed radian/sec

2π radians in full circle

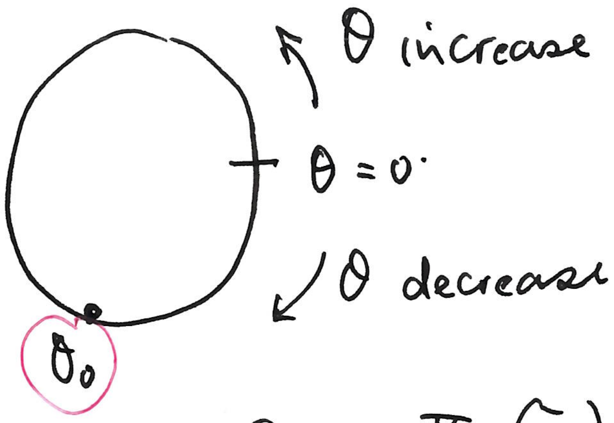
2π rads $= 360°$

what dirn on the circle?

clockwise and anticlockwise ??

Simplest dynamic on the circle is $\boxed{\dot{\theta} = \frac{d\theta}{dt} = \omega}$, a constant.

ω large and positive (fast anticlockwise spin)



$\omega > 0$ $\quad \frac{d\theta}{dt} > 0$ $\quad \theta$ increases with time. (anti-clockwise)

$\omega < 0$ $\quad \frac{d\theta}{dt} < 0$ $\quad \theta$ decreases with time

$$\theta = -\frac{\pi}{2} \; (\equiv) \; \theta = \frac{3\pi}{2}$$

ω is a constant (cf. $\dot{x} = k$, k constant)

For $\mathbb{R}$ we had $\dot{x} = f(x)$ (in general) $f : \mathbb{R} \to \mathbb{R}$.

S we need $\dot{\theta} = g(\theta)$ $\quad g : S^1 \to S^1$

To have a coherent function g

g needs to satisfy $\underbrace{g(\theta + 2\pi) = g(\theta)}_{\text{Same point.}}$ IMPORTANT.

$g(\theta + 2k\pi) = g(\theta)$
$k \in$ integers

9.3

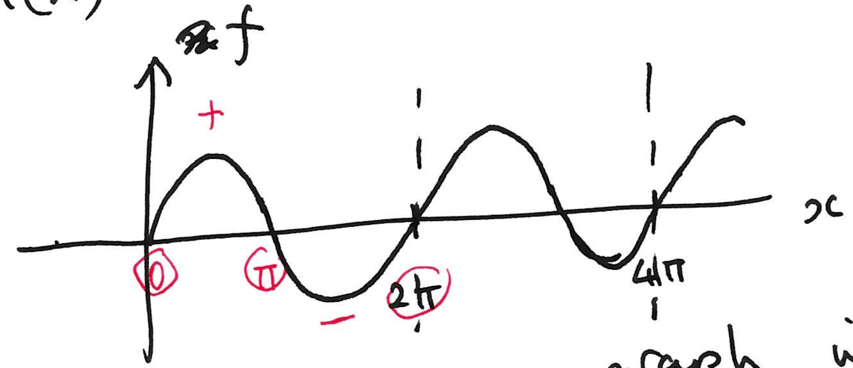
We need to use trigonometric functions which periodicity of 2π $(2k\pi)$

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x + 2k\pi) \equiv f(x)$, $k \in \mathbb{Z}$

$$\frac{df}{dx}(x + 2k\pi) = \frac{df}{dx}(x)$$

$\uparrow$ constant

$\mathbb{R}$ $\quad f(x) = \sin(x)$



translate one graph in $[0, 2\pi)$ to same graph in $[2\pi, 4\pi) \Rightarrow$ etc

$\downarrow$ $\qquad \downarrow$

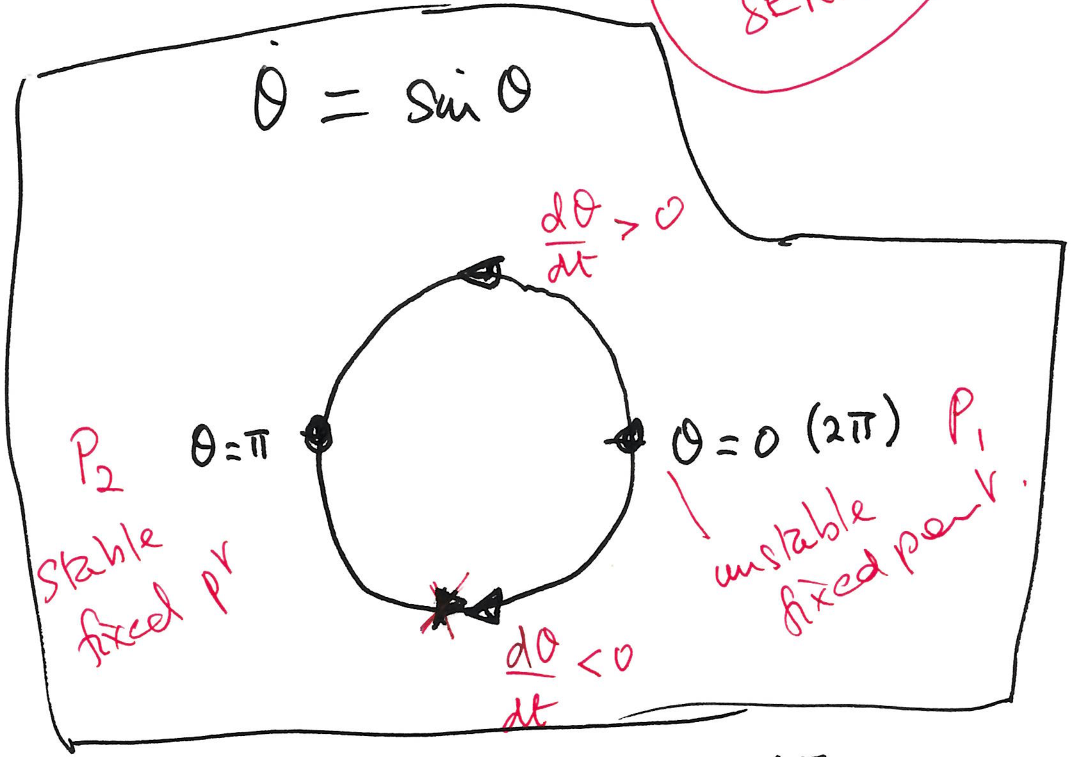
S $\quad \dot{\theta} = \sin(\theta)$ makes sense for $\theta \in S$

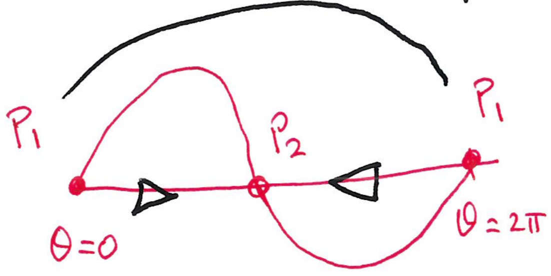
$\theta = x \pmod{2\pi}$

Page 19 $\quad f(x) = \sum_{k \in \mathbb{I}} a_k \cos(kx) + b_k \sin(kx) \quad \mathbb{I} \subseteq \mathbb{Z}$ 9.4

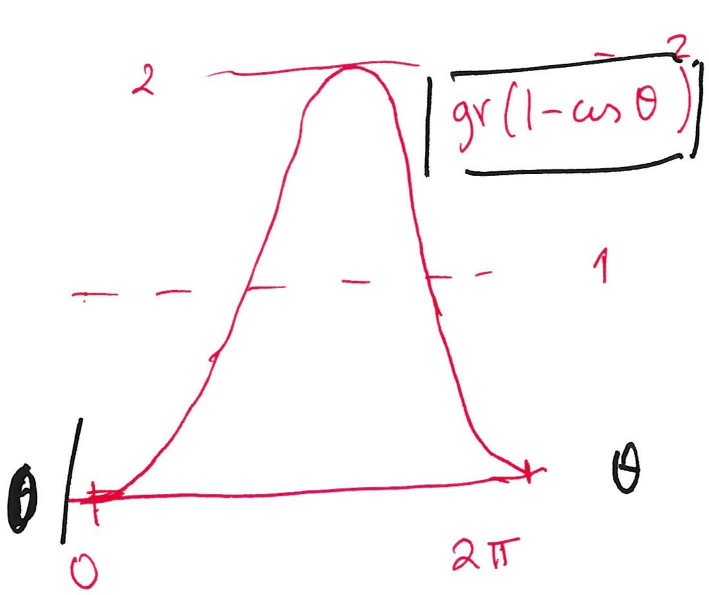
FOURIER SERIES

$\dot{\theta} = \sin\theta$

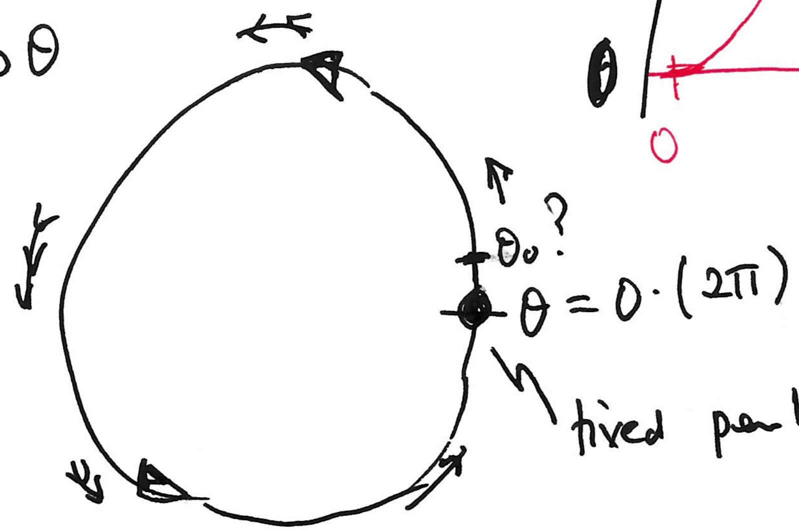




P_1, P_2 are the zeroes of $\sin\theta$



$\dot{\theta} = 1 - \cos\theta$

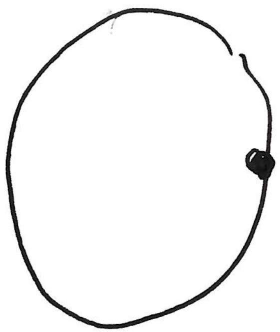


1 fixed pt.

fixed point $\dot{\theta} > 0$ otherwise

Different from what we saw on the real line

drift from A $\rightarrow$ towards B



0 $(\theta = 0)$ drift from 0 towards 0 again

More general example 3.2

$$\dot{\theta} = \omega - a\sin\theta$$

(cf $\dot{\theta} = 1 - \cos\theta$; $\dot{\theta} = 1 - \sin\theta$)

FP: $\theta = 0$; FP: $\theta = \pi/2$

$$\int \frac{d\theta}{\omega - a\sin\theta} = \int dt$$

$u = \tan \theta/2$ $\quad \frac{du}{d\theta} = \sec^2\left(\frac{\theta}{2}\right) \cdot \frac{1}{2}$

$\sin\theta = \frac{2u}{1+u^2}$ Substitution

$$T = \int_0^T dt = \int_{-\infty}^{+\infty} \frac{du}{wu^2 - 2au + w} = \int_{-\infty}^{+\infty} \frac{du}{w\left(u - \frac{a}{w}\right)^2 + w - \frac{wa^2}{w^2}}$$

$$= \frac{1}{w} \int \frac{du}{\left(u - \frac{a}{w}\right)^2 + \left(1 - \frac{w^2}{a^2}\right)}$$

$$\int \frac{dy}{y^2 + b^2} = \left[\frac{1}{b} \tan^{-1}\left(\frac{y}{b}\right)\right].$$

$$= \frac{2\pi}{\sqrt{w^2 - a^2}}$$

Note $a \to w^-$ $\quad \underline{\underline{T \to \infty}}$

$$\dot{\theta} = w - a \sin\theta$$

~~th~~ <u>can we see that dynamically?</u>

Lecture. 10

$$\dot{\theta} = \omega - a\sin\theta$$

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$a, \omega > 0$

$\sqrt{\omega^2 - a^2} \to 0$ $\underline{\underline{a \to \omega^-}}$ & $T \to \infty$

$a \to 0$

$\sqrt{\omega^2 - a^2} \to \omega$ $\quad T = \frac{2\pi}{\omega}$.

$a \to 0$ $\quad \dot{\theta} = \omega$, constant rotation at any speed ω through 2π radians

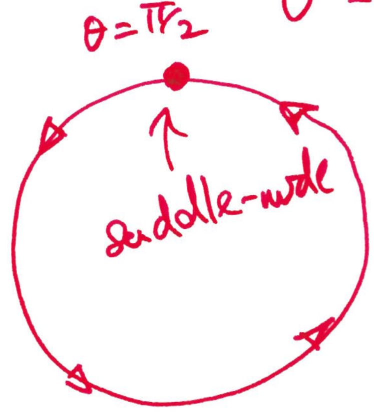
$$T = \frac{2\pi}{\omega}$$

$a \to \omega^-$

$$\dot{\theta} = \omega(1 - \sin\theta)$$

$\dot{\theta} = 1 - \sin\theta$ $\quad$ FP $\theta = \pi/2$

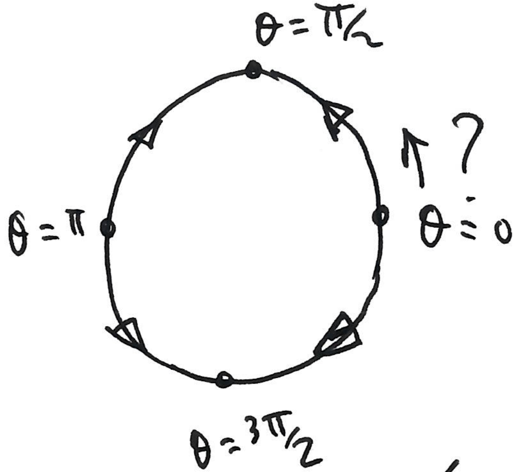
$\dot{\theta} > 0$



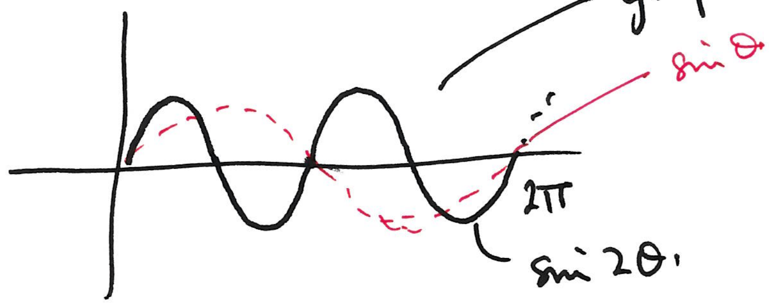
recall similar to

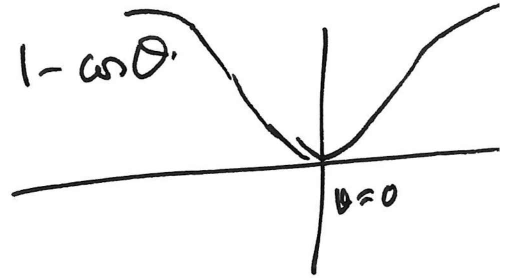
$$\dot{\theta} = 1 - \cos\theta$$

10.2.

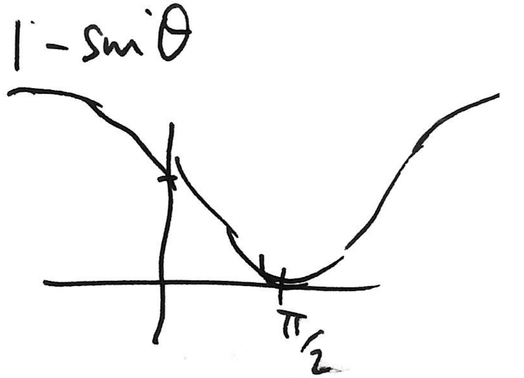


$\dot{\theta} = f(\theta)$ $\quad f$?

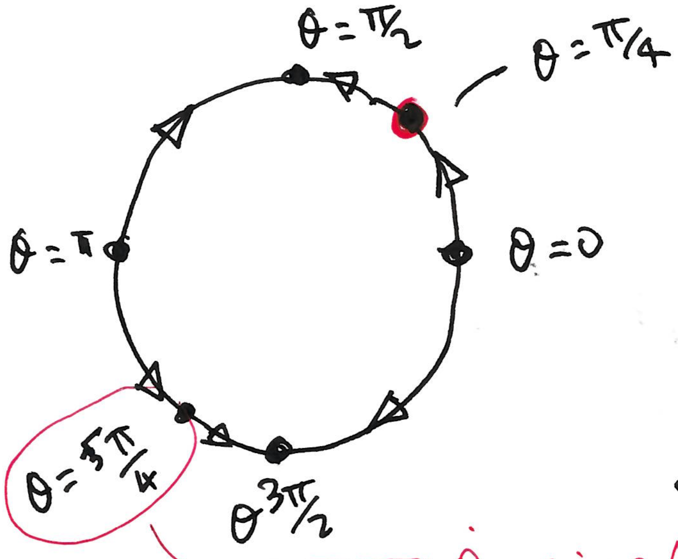




$f(\theta) = \sin 2\theta$



$\dot{\theta} = \sin 2\theta$



$\dot{\theta} = \sin 2\theta \left(1 - \cos\left(\theta - \frac{\pi}{4}\right)\right)$

> 0 except for $\theta = \frac{\pi}{4}$

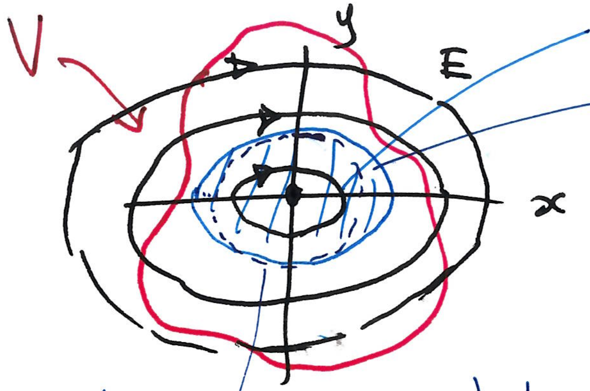
$= 0 \quad \theta = \frac{\pi}{4}$

$\operatorname{sgn}(\sin 2\theta) = \operatorname{sgn}\left(\sin 2\theta \left(1 - \cos\left(\theta - \frac{\pi}{4}\right)\right)\right)$ except at $\theta = \frac{\pi}{4}$

$\dot{\theta} = \sin 2\theta \left(1 - \cos\left(\theta - \frac{\pi}{4}\right)\right)\left(1 - \cos\left(\theta - \frac{5\pi}{4}\right)\right)$

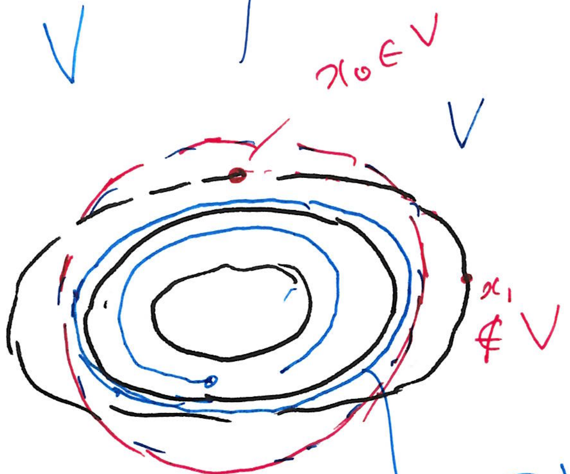
10.3

Stability $\qquad \dot{x} = y, \quad \dot{y} = 2x$



stability choose V, can I choose a nbd $U \subseteq V$ such all solns curves through U remain in V

Starting in U?? to stay in V



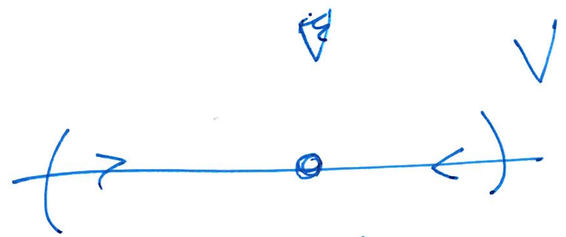
U?

$U = V$ does not work.

$U \subset V$

↑ usually U is smaller than V

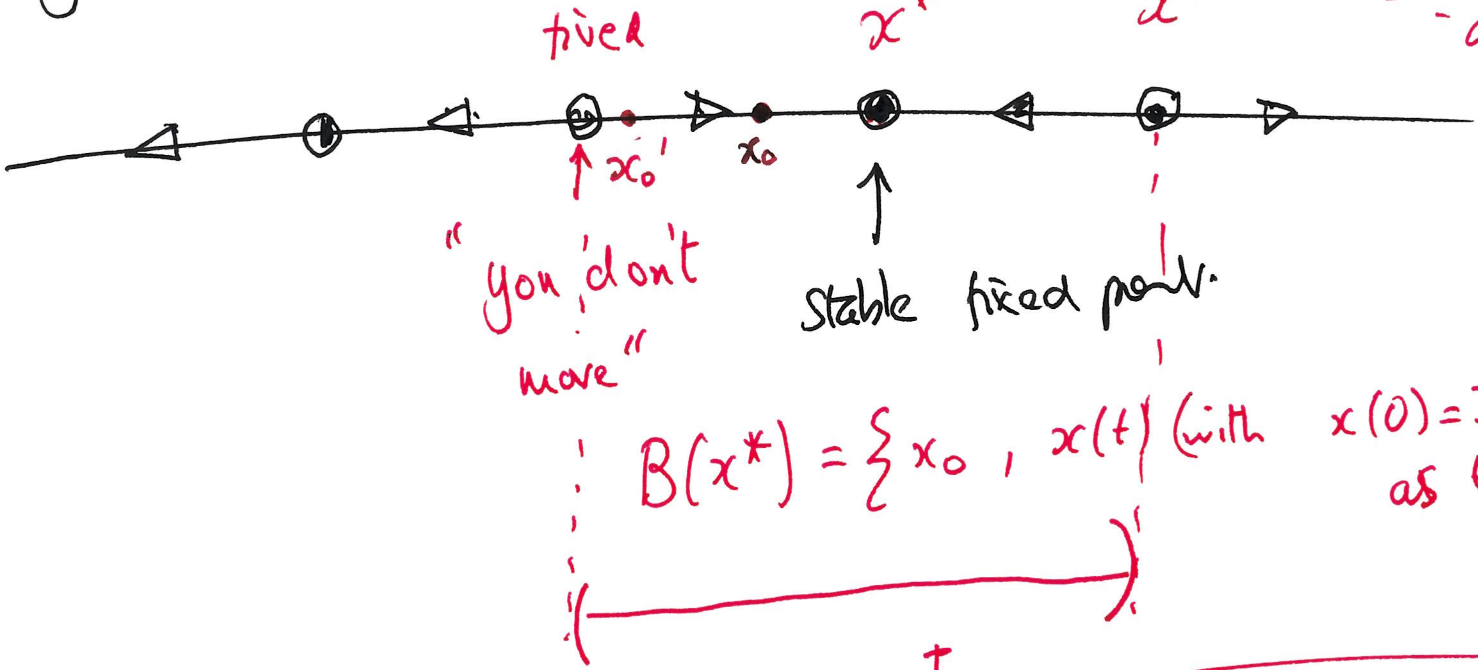
$U \neq V$.



$U = V$ every time works at a stable point.

Basin of attraction of fixed point.

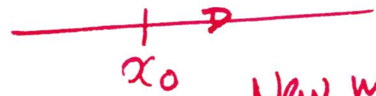
$\varphi_t(x_0) \to x^*$ as $t \to \infty$



$B(x^*) = \{x_0,\ x(t)$ (with $x(0) = x_0) \to x^*$ as $t \to \infty$

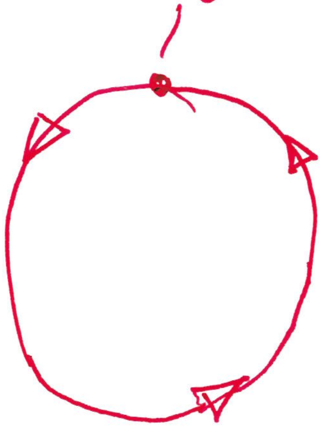
$B(x^+) = \{x_0^+\}$.

$\boxed{x(t),\ \text{where } x(0) = x_0}$



New notation

$\varphi_t(x_0)$ = pt after time t that x_0 has moved to.



just one fixed pt

$B(\theta^*) = S^1$