LECTURE 7

Further comments on super- sub criticality

7.1 1

Pitchfork bifurcation

$$\begin{array}{ccc} & C & E \\ \dot{x} = & rx & - x^3 \end{array} \quad — ①$$

$$\dot{x} = rx + x^3 \quad ②$$

$A=0 \quad \frac{\partial}{\partial r} \qquad B=0 \quad \frac{\partial^2}{\partial x^2} \qquad E = -1, \ +1 \qquad C \neq 0 \ (=1)$

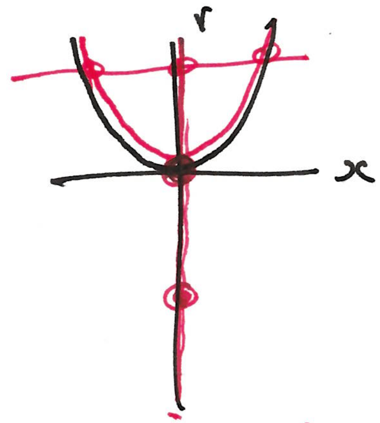
FPs $\quad \dot{x} = x(r - x^2) \quad ①$

$x \equiv 0, \ \forall r \qquad x = \pm\sqrt{r}$

3 fixed $r>0$

1 $r=0$

1 $r<0$



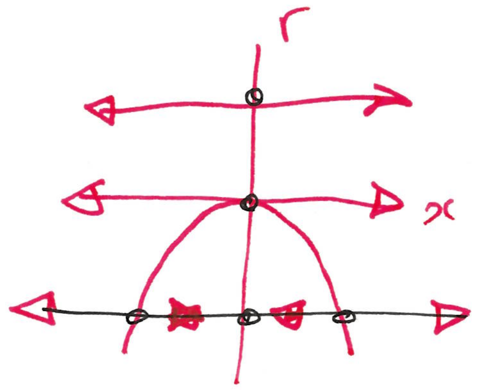
Super critical

$r=0$ bifurcation parameter value

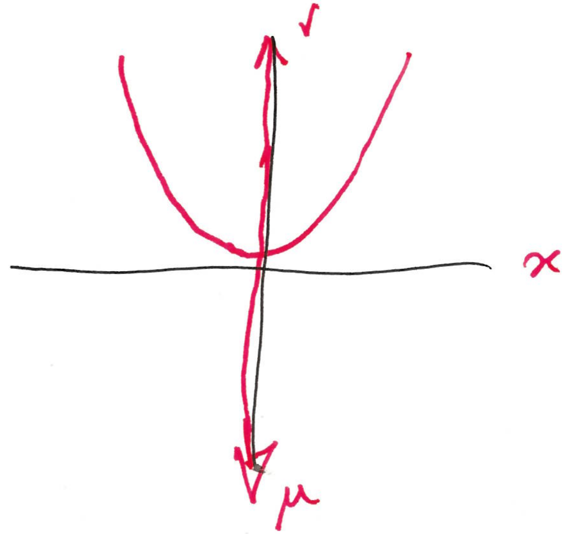
7.2a

2

$\dot{x} = rx + x^3$



subcritical
multiple # of FP occurs
for $r < 0$.



supercritical

$\dot{x} = -rx + x^3$

$= x(-r + x^2)$

$x \equiv 0 \quad \forall r$

$-r + x^2 \equiv r = x^2$

Let $\mu = -r \Rightarrow \dot{x} = \mu x + x^3 \rightarrow$ subcritical w.r.t. μ

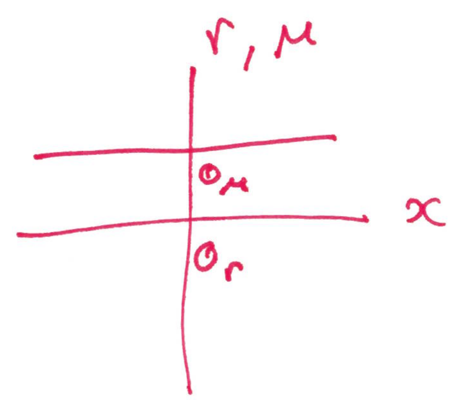
7.2b

$$\dot{x} = rx - x + x^3$$
$$= (r-1)x + x^3$$

Let $\mu = r - 1$ $\quad \dot{x} = \mu x + x^3$

subcritical at $\mu = 0$

Subcritical at $r = 1$



Having identified the bifurcation in new coordinates $(x, r) \to (y, \mu)$

Also for saddle node super and sub? in the original coordinates

3.4.1 $\quad \dot{x} = rx + 4x^3$

3.4.2 $\quad \dot{x} = rx - \sinh(x) \quad (= \sinh x - rx).$

3.4.4 $\quad \dot{x} = x + \dfrac{rx}{1+x^2}$

7.3

3.4.5

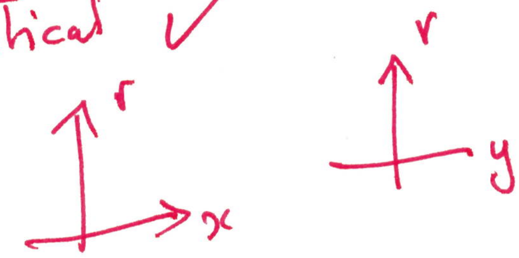
3.4.1 $\quad \dot{x} = rx + 4x^3$ Subcritical

$x = \alpha y, \quad \alpha \neq 0$

$\dot{x} = \alpha \dot{y} = r\alpha \cancel{y} + 4\alpha^3 y^3$

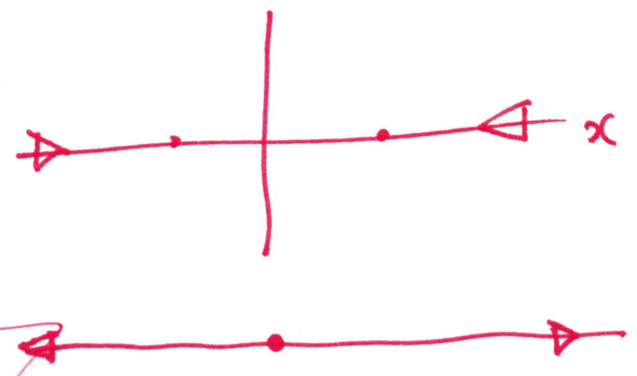
$\dot{y} = ry + 4\alpha^2 y^3 \qquad \alpha^2 = \frac{1}{4}, \ \alpha = \frac{1}{2}$

$\dot{y} = ry + y^3 \qquad$ Subcritical ✓



Question from class

$\dot{x} = rx - x^3$



$-x^3$

$+x^3$

7.4

3.4.2 $\dot{x} = rx - \sinh(x)$

$$\dot{x} = rx - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right)$$

$$= (r-1)x - \frac{x^3}{3!} + \text{H.O.T.}$$

$\mu = r-1$

$$\dot{x} \quad \mu x - \frac{x^3}{3!}$$

Supercritical in $\mu = 0$

$\Updownarrow$

Supercritical in r at $r = 1$

$$\dot{x} = x + \frac{rx}{1+x^2} = x + rx(1 - x^2 + x^4 \cdots)$$

$$= x(r+1) - rx^3$$

$\mu = r+1$ $= x\mu - (\mu-1)x^3 = x\mu + x^3 + \cancel{\mu x^3}$

> decide criticality

& bifurcation value $r=-1$

3.4.9. $\dot{x} = x + \tanh(rx) = f(x)$ 7.5

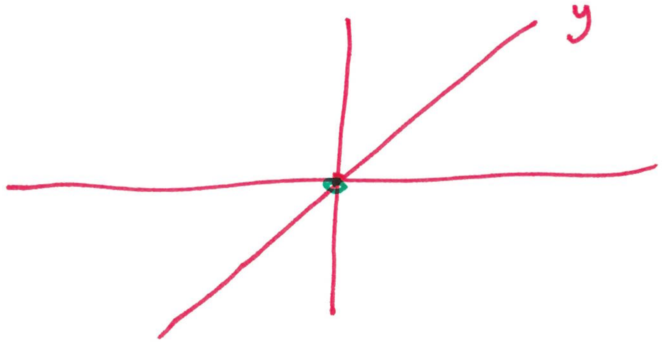
$\downarrow$ Slope

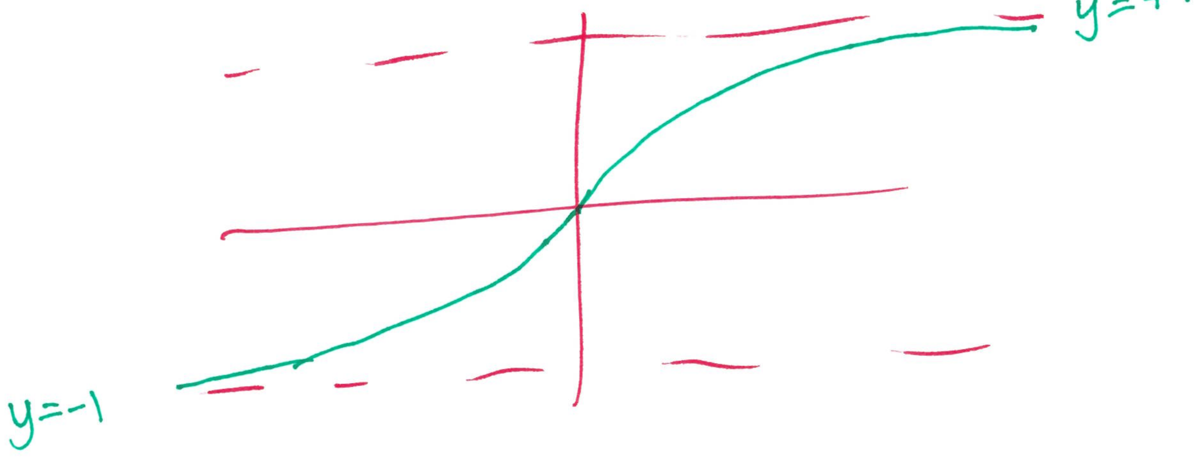
$\tanh'(x) \xrightarrow{D} \operatorname{sech}^2(x)$

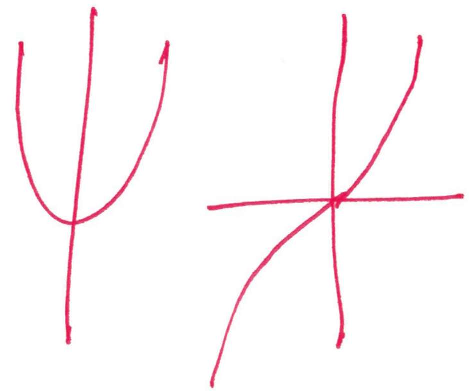
$x = 0 \Rightarrow f(x)$ FP

$f'(x) = 1 + r \operatorname{sech}^2(rx)$ $x = 0$?

$= 1 + r$







NOTE

$r = -1$ gives

$f(0) = 0$

$f'(0) = 0$.

Bifurcation pt?

7.6

$r = -1$ gives slope 0.

So check ($r = -3/2$, $r = -\frac{1}{2}$) the graphs

of $f(x) = x + \tanh(rx)$, FP regime?

$r = -3/2 \longrightarrow r = -1, \longrightarrow r = -\frac{1}{2}$.

Power series of $\tanh x = \dfrac{\sinh(x)}{\cosh(x)}$

$$= \frac{x + \frac{x^3}{3!} + \cancel{\frac{x^5}{5!}}}{1 + \frac{x^2}{2!} + \cancel{\frac{x^4}{4!}}} = x + \dots ?$$

$\downarrow$

$\tanh(rx)$

Determination of bifurcation type

LECTURE 8 8.1

2.5 $\dot{x} = f(x, r)$

t = independent variable

r = parameter.

x: dynamic dependent variable

Bifurcations occur when phase portraits in x change structure as r increases. The structure of f.ps changes at such a critical parameter, $r = r_c$

FPs $f_r(x) = 0$ $\quad (f(x,r) = 0)$

Recall 2.5.1. Implicit Function Theorem.

$$f(x, r) = 0 \quad \Rightarrow \quad x = x(r)\ ?$$

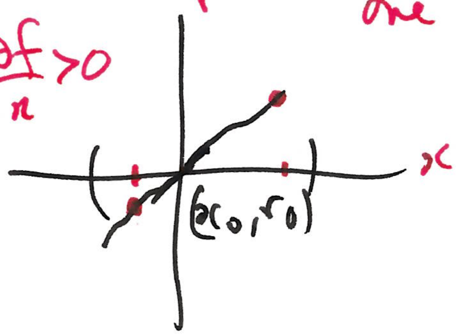
$\dfrac{\partial f}{\partial x} \neq 0$ at (x_0, r_0), then there is solution

$$x = x(r) \quad \text{s.t.} \quad f(x(r), r) \equiv 0$$

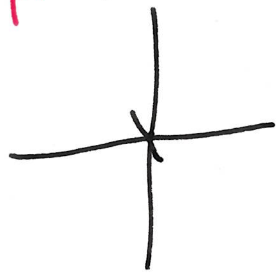
in a nbd of (x_0, r_0)

8.2

$\frac{\partial f}{\partial x} > 0$



one fixed pt



$\frac{\partial f}{\partial x} < 0$

No change in the number of fixed pnts

$\rightarrow$ no bifurcation

$\frac{\partial f}{\partial x} \neq 0$

Bifurcations can only occur if $f(x, r) = 0$ **, $\frac{\partial f}{\partial x}(x, r) = 0$ *

$x = x^*$, $r = r_c$ where $f(x^*, r_c) = 0$ $\frac{\partial f}{\partial x}(x^*, r_c) = 0$.

$$f(x, r) = \cancel{f(x^*, r_c)}^{**} + \cancel{\frac{\partial f}{\partial x}(x^*, r_c)}^{*}(x - x^*) + \frac{\partial f}{\partial r}(x^*, r_c)(r - r_c) + \dots$$

(cf 2.35)

Expand as a Taylor Exp at $(x, r) = (x^*, r_c)$

1 dim $\quad f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \dots$

8.3

Saddle-node

$$\dot{y} = A\mu + By^2 + Cy\mu$$

$$= A\mu + B\left(y + \frac{C\mu}{2B}\right)^2 - B\left(\frac{C\mu}{2B}\right)^2$$

$$z = y + \frac{C\mu}{2B} \qquad \dot{z} = \dot{y} = A\mu + Bz^2 - C\mu^2$$

$$\nu = \mu - \frac{C\mu^2}{A}$$

$$\dot{z} = A\nu + Bz^2$$

$$\dot{w} = \eta + w^2$$

C is not important if $A, B \neq 0$

Transcritical

$A = 0 \qquad \mu = (r = r_c) = 0$

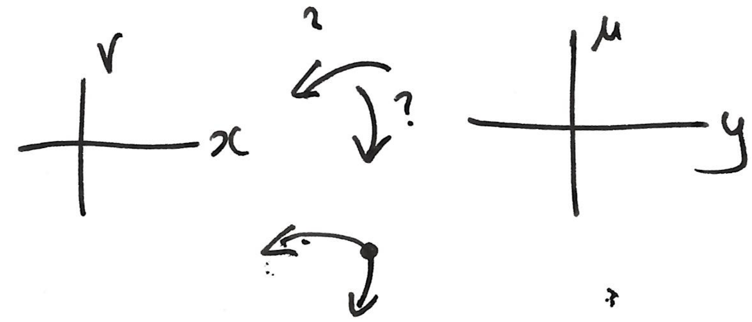
$B \neq 0,\ C \neq 0.$

$$\dot{y} = \mu y + y^2$$

Pitch fork $A = 0,\ B = 0,\ C \neq 0,\ E \neq 0$ $\left(\frac{\partial^3 f}{\partial x^3}(x^*, r_c) \neq 0 \right)$ 8.4

$$\dot{x} = rx + x^3$$

$$\dot{y} = \mu y \pm y^3$$

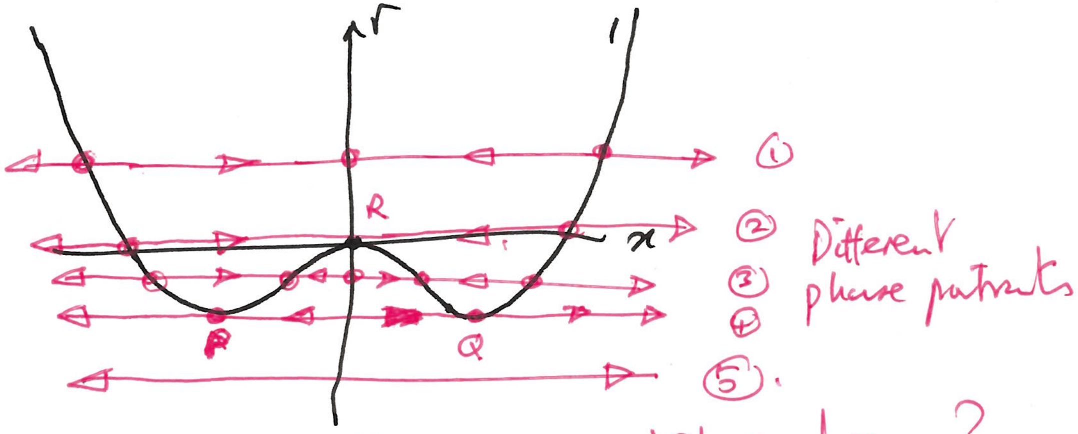


You need to "track back" from the (y, μ) information to the behaviour (x, r)

8.5

$\dot{x} = x^5 - x^3 - rx$

FPs $\quad x^5 - x^3 - rx = 0 \quad \begin{cases} r = x^4 - x^2 & \approx x^4, \ |x| \text{ large} \quad \approx -x^2 \ |x| \text{ small.} \\ x \equiv 0 & x=0,\ x=\pm 1 \end{cases}$



point P saddle-node bifurcation ?

point Q saddle-node bifurcation ?

point R subcritical pitchfork

Find coordinates of P, Q (R = (0,0)!).

Expand in local coordinates at each point (R already done!)

Identify bifurcations.

Quiz 1 8.6

$$\dot{x} = x^4 - 2\alpha x^2 + 1$$

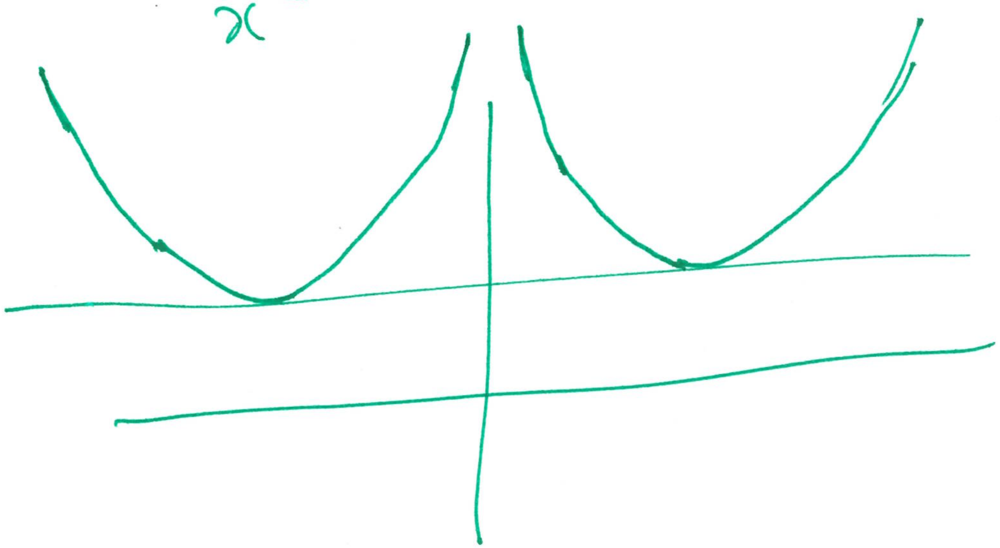
FP's

$$x^2 = \alpha \pm \sqrt{\alpha^2 - 1}$$

Comment re.
Quiz one question.

$$2\alpha x^2 = x^4 + 1$$

$$\cancel{2}\alpha = \frac{x^4+1}{x^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$$



$\alpha = 1$