Week 3 Lectures 5–6. Bifurcation Theory

$\dot{x} = f(x) = x^2 - 3x + 2$ Chapter 1. single phase portrait

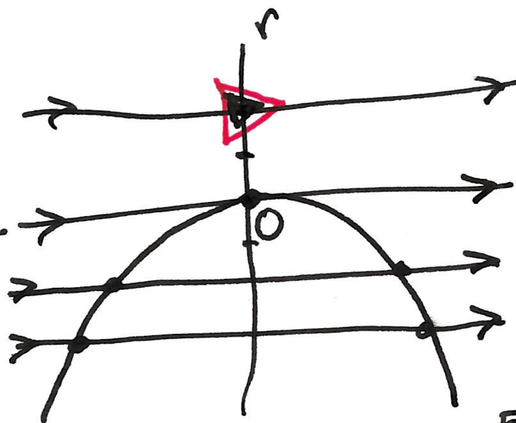
$\dot{x} = f_r(x) = \underline{r + x^2}$, $3r - x^3$

phase portrait for every $r \in \mathbb{R}$.

lots of them could be the same qualitatively

$\dot{x} = r + x^2$ (p10) FPs $r + x^2 = 0$ $x = \pm\sqrt{-r}$, $r \leq 0$

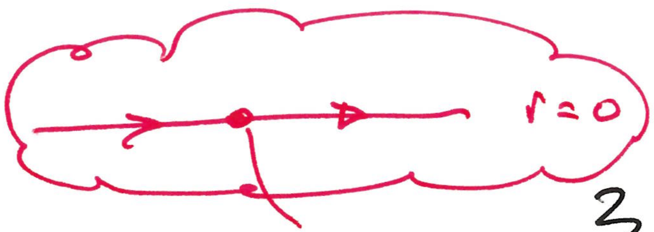
Bifurcation diagram



① FPs $r > 0$ – $\underline{\underline{0}}$

② $r = 0$ – $\underline{\underline{1}}$

③ $r < 0$ – $\underline{\underline{2}}$



saddle-node unstable fixed pt

3 topological types of phase

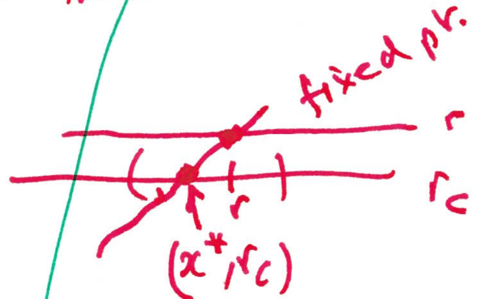
Saddle-node bifurcation

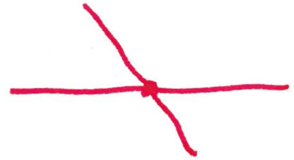
$\dot{x} = r + x^2$, $x^* = 0$, $r_c = 0$. $f(x^*, r_c) = 0$

Ref 2.5.1 $\frac{\partial f}{\partial x}(x^*, r_c) \neq 0$. what does f look like

> 0

Implicit value theorem $\Rightarrow$ $x = x(r)$ s.t. $x = x^*$, $r = r_c$.





unique fixed point $f(x, r)$ in nbd of $r = r_c$

$\therefore$ No bifurcation can occur.

$f(x^*, r_c) = 0$, $\frac{\partial f}{\partial x}(x^*, r_c) = 0$, conditions needed for a potential bifurcation

Ex $x^2 + r = 0$ $\quad 2x = 0$ $\quad x^* = 0$, $r_c = 0$

$B = 1$ $A = 1$

Ex 2.3 $\dot{x} = r - x - e^{-x}$

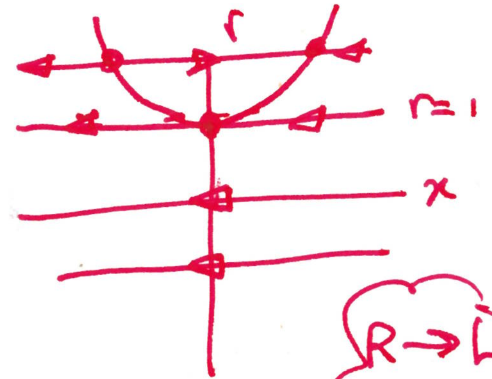
$$\dot{x} = r - x - \left(1 - x + \frac{x^2}{2!} + \cdots\right) \quad \text{H.O.T.}$$

R→L

$$= (r-1) - \frac{x^2}{2} + \text{HOT}$$

(5.3)

$x^* = 0,\ r_c = 1$



R→L

For small x $\quad x \approx \pm 2\sqrt{r-1}$,

2	$r>1$
1	$r=1$
0	$r<1$

Let $\mu = r-1$

Example of coordinate transformation

$$\dot{x} = \mu - \frac{x^2}{2} \qquad \text{cf.} \qquad \boxed{\dot{x} = r + x^2}$$

Note

$\alpha x = y \qquad \dot{y} = \alpha\dot{x} = \alpha\mu - \cancel{\frac{\alpha x^2}{2}} \Rightarrow \dot{y} = \alpha\mu - \frac{y^2}{2\alpha}$

$y = -\frac{1}{2}x$

$\dot{y}>0 \Leftrightarrow \dot{x}<0$

$\dot{y}<0 \Leftrightarrow \dot{x}>0$

$\dot{y}=0 \Leftrightarrow \dot{x}=0$

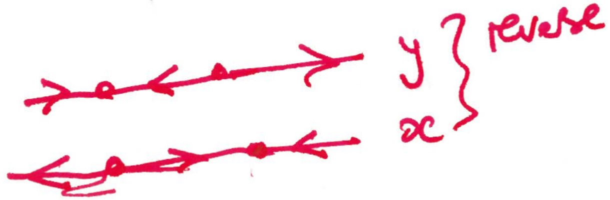
Let $\alpha = -\frac{1}{2} \qquad \dot{y} = -\frac{1}{2}\mu + y^2 \qquad$ Let $\nu = -\frac{1}{2}\mu$

$$\boxed{\dot{y} = \nu + y^2}$$

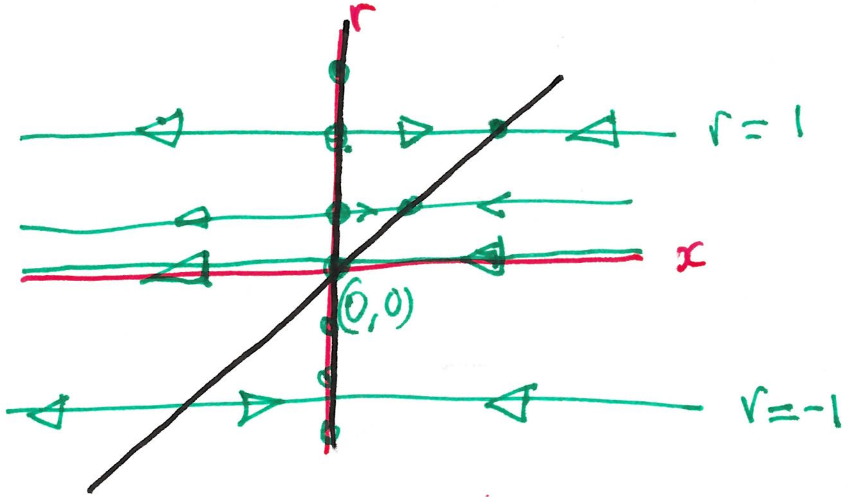
change of variable y x

& also μ

$$\dot{x} = \alpha\mu + \beta x^2 \Longleftrightarrow \dot{y} = \nu + y^2$$



2.4 $\dot{x} = \underset{f_r(x)}{rx - x^2}$ FP $rx - x^2 = 0$ $x(r-x) = 0$ 5.4



"transcritical bifurcation."

$x = 0$, $r = 1$ $r > 0$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=0 \\ r=1}} = r - 2x = r - 0 = r > 0$$

unstable

$x = r$, $r = 1$ $r > 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=1 \\ r=1}} = r - 2r = -1$$

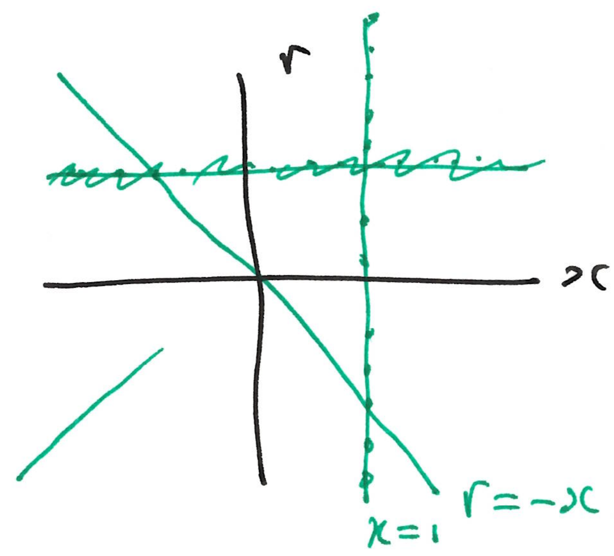
stable.

Similar calc for $r < 0$.

Ex 2.6. $f_r(x) = f(x,r) = 0$

$\dot{x} = r\ln(x) + x - 1$ FPs Observation gives a fixed pt (6.1)

$x=1$ is a fixed point. $\forall r \in \mathbb{R}$



$f_r(x) = 0$ $f(x,r) = 0$

$\frac{\partial f_r(x)}{\partial x}$ $\frac{\partial}{\partial x} f(x,r) = 0$

$$\frac{\partial}{\partial x}\left(f(x,r)\right) = \frac{r}{x} + 1 = 0$$

$$r = -x.$$

$x^* = 1$, $r_c = -1$

$y = x - 1$ $(= x - x^*)$ in y, μ coordi $\mu = 0, y = 0$ at the

$\mu = r + 1$ $(= r - r_c)$ pt of interest $x^* = 1$, $r_c = -1$

6.2

$y = x - 1$

$\mu = r + 1$

$$\dot{x} = \dot{y} = r\ln(1+y) + y + 1 - 1$$

$$= r\left(y - \frac{y^2}{2} + \frac{y^3}{3} + \cdots\right) + y = (r+1)y - \frac{ry^2}{2} + \text{H.O.T.}$$

$$= \mu y - \frac{ry^2}{2} = \mu y - (\mu - 1)\frac{y^2}{2} = \mu y + \frac{y^2}{2} + \text{other terms}$$

$r = \mu - 1$

$C = 1 \quad B = \frac{1}{2} \quad A = 0$

$\Downarrow$

Trans. B.

transcritical bifurcation at $x^* = 1$, $r_c = -1$

Pitchfork bifurcation

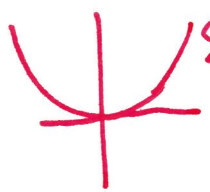
$C \neq 0,\ B = 0,\ A = 0$ the only term up to 2nd order is

$$\dot{x} = C\mu x + Ex^3$$

Simple case $C = E = 1$

$$\dot{x} = \mu x + x^3$$

supercritical

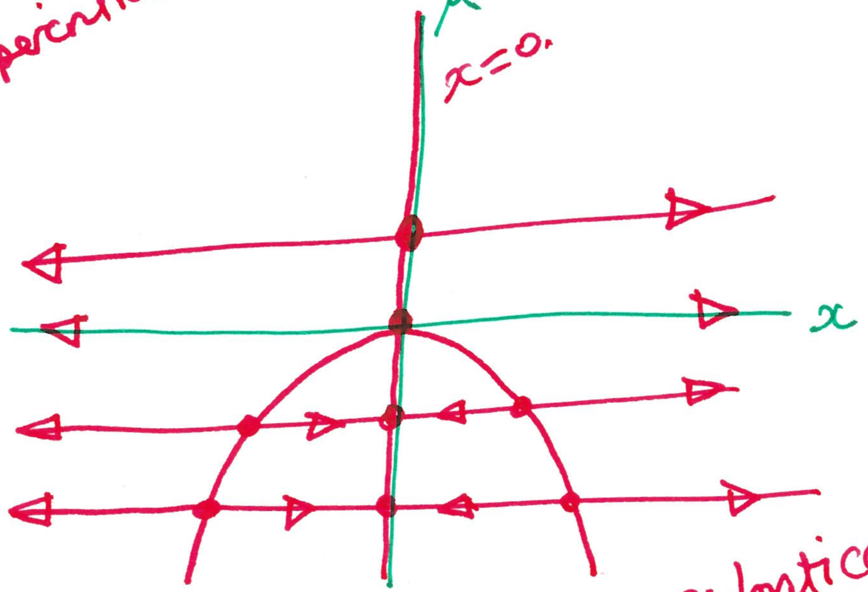


FPs $\mu x + x^3 = 0$

$x(\mu + x^2) = 0 \ \left(= (x-0)(x-\sqrt{-\mu})(x+\sqrt{-\mu})\right)$

$x = 0$

$\mu + x^2 = 0$



Subcritical Pitchfork Bif.

pitchfork bifurcation.

$\mu > 0$	1 FP
$\mu = 0$	1 FP
$\mu < 0$	3 FP.

6.4

Ex $\dot{x} = \underbrace{\mu x + \frac{x}{1+x^2}}_{f(\mu,x)}$

FP $x=0$

$\frac{\partial f}{\partial x} = 0. \Rightarrow \mu + \frac{(1+x^2).1 - x.2x}{(1+x^2)^2} = 0$

$\mu + \frac{1-2x^2}{(1+x^2)^2} = 0$

$\mu + 1 = 0$ $\mu = -1$

$x^* = 0, \quad \mu_c = -1$

$x = x - 0 \to x^*$

$\nu = \mu + 1 \to \mu_c$

local coords

Taylor expn.:

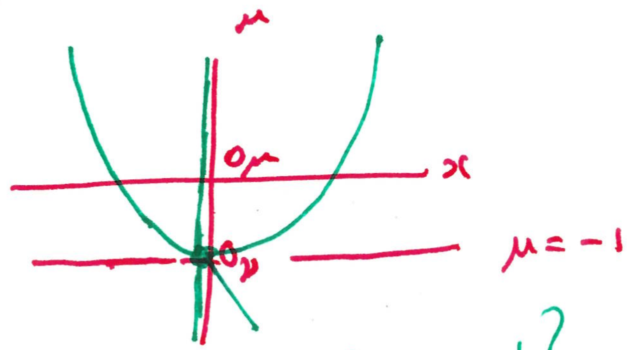
$(x,\mu) \to (x,\nu)$

$\dot{x} = (\nu-1)x + x(1+x^2)^{-1}$

$= (\nu-1)x + x(1-x^2) + \text{H.O.T.} = \nu x - \cancel{x} + x - x^3 + \text{h.o.t}$

$$\boxed{\dot{x} = \nu x - x^3 + \text{H.O.T.}}$$

PF Bifurcation



FP $\nu x - x^3 = 0$

$x(\nu - x^2) = 0$

$\left.\begin{array}{l} x=0, \mu=-1 \\ \Leftrightarrow x=0, \nu=0 \end{array}\right\}$

$C \neq 0, \; E \neq 0.$