Lecture 3

3.1 Population

$N \geq 0$

Example 1.6.

$\dot{N} = rN \rightarrow$ exponential growth $N(t) = N_0 e^{rt}$

$N = N_0$, $t = 0$.

$\dot{N} = rN\left(1 - \frac{N}{k}\right) = \alpha N - \beta N^2$ logistic correction

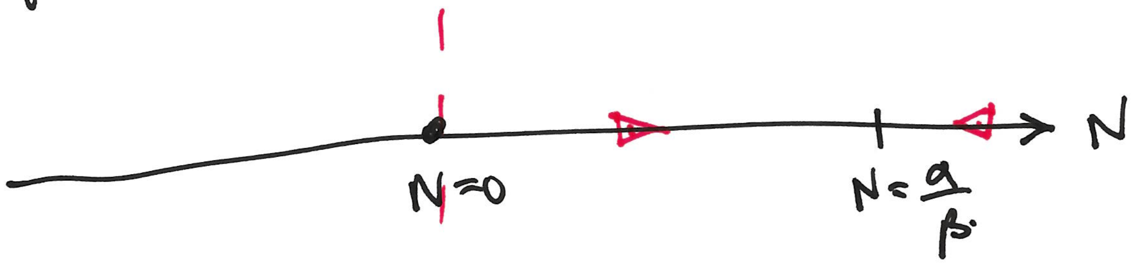
$\alpha = r$, $\beta = r/k$.

parameters $r, k > 0 \Rightarrow \alpha, \beta > 0$

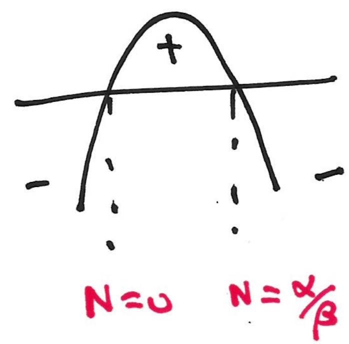
$\dot{N} = \alpha N - \beta N^2 = N(\alpha - \beta N) \rightarrow$ phase portrait?

$N = 0 \rightarrow$ fixed pt. $\quad N = \alpha/\beta > 0$

$f(N) = \alpha N - \beta N^2$



unstable equilibrium.



3.2

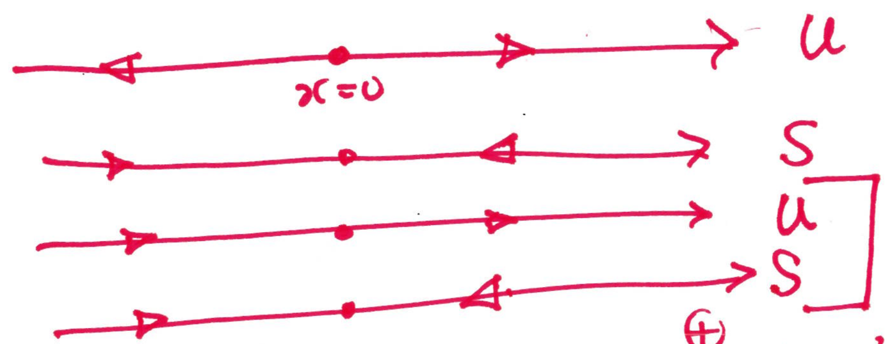
Linear stability at a fixed point

(i) $\dot{x} = x$ $\left(= f(x)\right)$

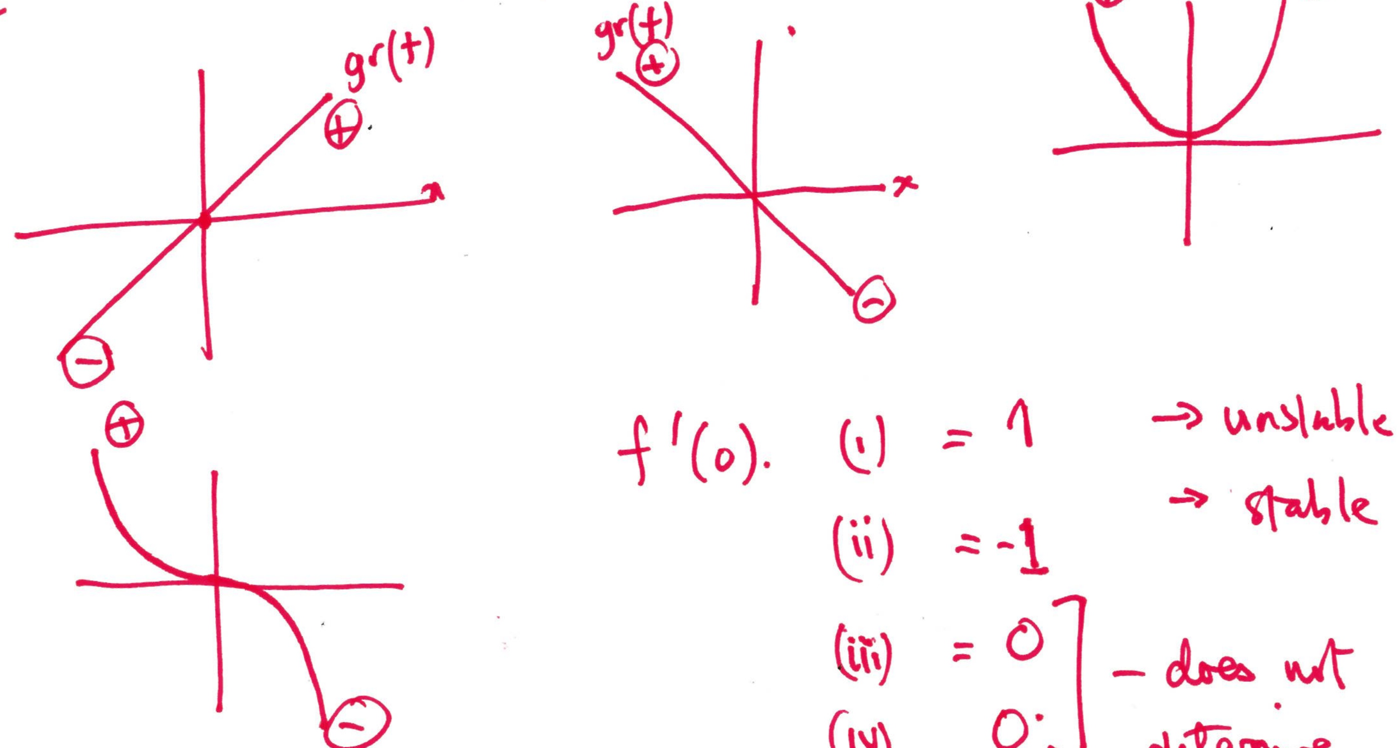
(ii) $\dot{x} = -x$

(iii) $\dot{x} = x^2$

(iv) $\dot{x} = -x^3$



gr(f) (i)



$f'(0)$. (i) $= 1$ → unstable

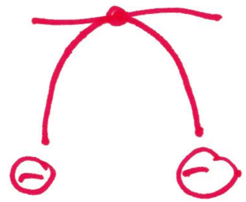
(ii) $= -1$ → stable

(iii) $= 0$

(iv) 0.

– does not determine stab/unstability

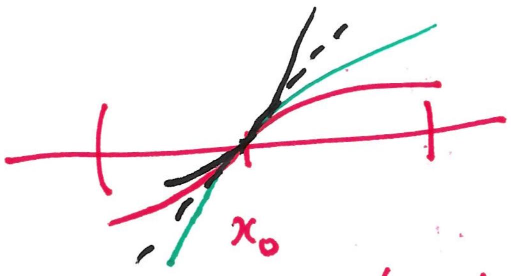
Others



3.3

$\boxed{\dot{x} = f(x)}$ $f(x)$, x_0 is a fixed point

$f'(x_0) > 0$ $(f(x_0) = 0)$



Neighborhood

f at $x = x_0$ goes from $-$ to $+$

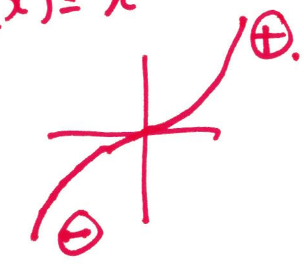
unstable

$f'(x_0) < 0 \longleftrightarrow$ stability

$\boxed{f'(x_0) = 0}$ – further investigation is needed

$\dot{x} = x^5$, $f(x) = x^5$ $f'(x) = 5x^4 = 0$ at $x = 0$

$x = 0$ F.P.



$\oplus$ $\ominus$

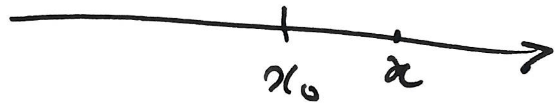
$\rightarrow$ unstable

$\dot{x} = x^6 + x^8$

$\oplus \rightarrow \oplus$

$x = f(x)$, x_0 fixed point.

$x = x_0 + \eta$



$$\dot{x} = 0 + \dot{\eta} = f(x_0 + \eta).$$

$$\overset{T.E.}{\equiv} \cancel{f(x_0)} + \eta f'(x_0) + O(\eta^2)$$

$$\therefore \dot{\eta} = \eta f'(x_0) + \overset{0}{\overset{\shortparallel}{\cancel{H.O.T.}}}$$

η small

Approximately a linear. system at $x = x_0$. For linear system

$f'(x_0) > 0$ unstable (linearly unstable)

$f'(x_0) < 0$ stable (linearly stable)

Examples show that $f'(x_0) = 0$ does not determine stability

Linear stability (at the ~~linear~~ part of $f(x)$ at $x_0 = x_0$)

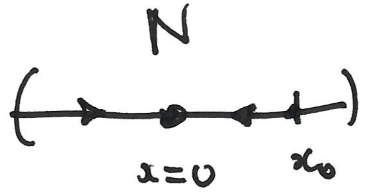
if $f'(x_0) < 0$

Linear instability if

$f'(x_0) > 0$

Stability

Asymptotic. $\dot{x} = -x$ $\rightarrow$



, $x(t) \in N$

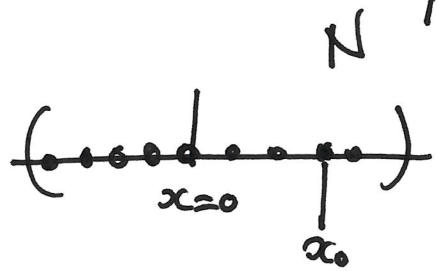
$x(t) \to 0$ as t incr

Asymptotic stability

~~Asymptotic~~

Non-asymptotic (just stable)

$\dot{x} = 0$ $\rightarrow$



$f(x) \doteq 0$

<u>all</u> fixed point

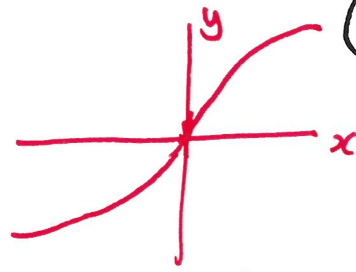
$x(t) \in N$

$x(t) \not\to 0$ as t incr.

$x=0$ is a stable pt / but not A.S.

(4.1)

Example 1.8 (p 8) $\quad \dot{x} = x^{1/3} \quad *$



$y = x^{1/3}$

$x = y^3$

Consider (1) ~~$x\equiv 0$~~ $\quad x(t) \equiv 0 \qquad \text{LHS} = 0 \ \forall t, \quad \text{RHS} = 0 \ \forall t$

$$\frac{dx}{dt}(t) \equiv 0$$

$\therefore \; x_1(t) \equiv 0$ is a soln of $*$.

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} x^{-2/3}$

slope infinite at $x = 0$

Consider (2) $\quad x(t) = \left(\frac{2}{3}t\right)^{3/2}$

$$\frac{dx}{dt}(t) = \frac{3}{2}\left(\frac{2}{3}t\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2}{3}t\right)^{1/2}$$

$$x^{1/3} = \left(\left(\frac{2}{3}t\right)^{3/2}\right)^{1/3} = \left(\frac{2}{3}t\right)^{1/2}$$

$x_2(t) = \left(\frac{2}{3}t\right)^{3/2}$ is a soln of $*$

$t = 0 \quad x_1(0) = 0, \quad x_2(0) = 0.$

same initial condition $x = 0, t = 0$

but 2 solutions.

Non-uniqueness

Existence & Uniqueness (Thm 1.1)

$\left.\begin{array}{l} x = x_0 \\ t = t_0 \end{array}\right\}$ $x(t)$? such that $x(t_0) = x_0$

$$\boxed{\dot{x} = f(x)}$$

$\dot{x} = f(x)$, f diff $x \in X = (a,b) \subseteq \mathbb{R}$

$\exists$ for $x_0 \in X$, $\exists$ $t \in I$

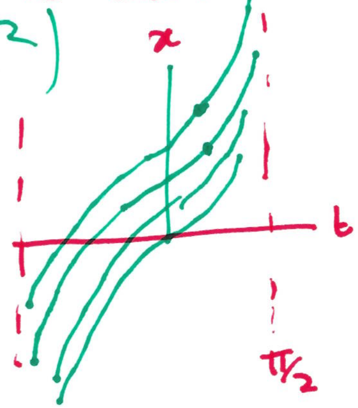
$I \ni 0$. (or t_0)

s.t. $x(t)$ exists as a soln of $\dot{x} = f(x)$ with $x(0) = x_0$

i.e. $x(t)$ exists locally for $x(t_0) = x_0$, $t = t_0$

Comment from student re exercise 1.5.1

$\dot{x} = (1 + x^2)$ $x(t) = \tan(t - t_0) + x_0$. $x(t_0) = x_0$



Bifurcations of ODEs on $\mathbb{R}$.

$\dot{r} = 0.$

$\dot{x} = r + x^2$, r is a parameter.

$f_r(x) = r + x^2$ fixed pt:

$f(x, r)$

$x^2 = -r$

$x = \pm\sqrt{-r}$ $r < 0$

$x = 0$ $r = 0$

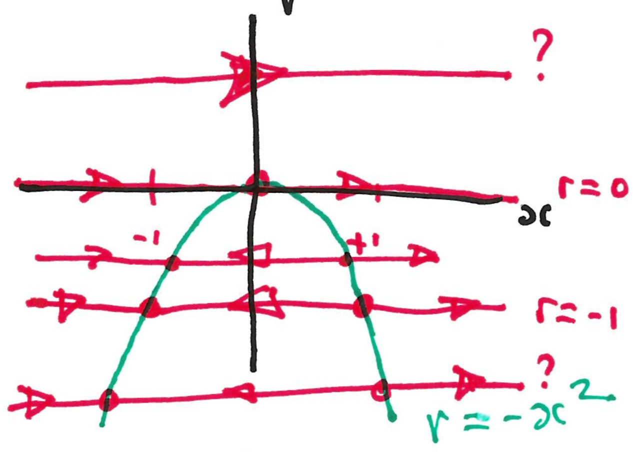
no FPs x? $r > 0$

dangerous description

* unstable/stable *

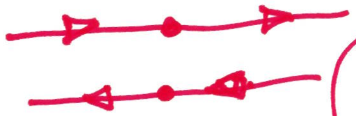
$r = 0$ $\dot{x} = x^2 : x = 0$

$r = -1$ $\dot{x} = -1 + x^2 : x = \pm 1$ qual same



"shunt"

"saddle-node" fixed point



both unstable!

bifurcation diagram.

example (canonical!) of a saddle-node bifurcation.

4.4

Bifurcation point $r = r_*$ is where the phase portrait of $\dot{x} = f(x,r)$ changes <u>topologically</u>