MTH744 January 2021 (Semester A)

Question 1

(a) (i) $\dot{x} = x^2(x^6 + x^3 - 1) \quad \left(= f(x)\right)$

Fixed points: $x^2(x^6 + x^3 - 1) = 0$

$\Rightarrow \quad x = 0$ and $x^6 + x^3 - 1 = 0$,

Now $x^6 + x^3 - 1 = 0 \Rightarrow (x^3)^2 + x^3 - 1 = 0 \Rightarrow \left(x^3 + \frac{1}{2}\right)^2 - \frac{5}{4} = 0$

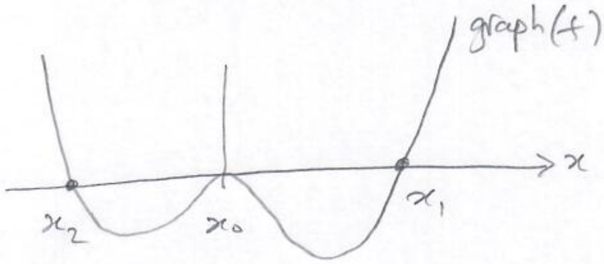
$\therefore$ Fixed points at $x_0 = 0$ & $x_1 = \sqrt[3]{\frac{\sqrt{5}}{2} - \frac{1}{2}}$, $x_2 = \sqrt[3]{-\frac{\sqrt{5}}{2} - \frac{1}{2}}$

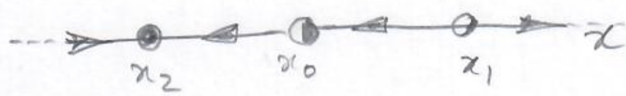
Note $\sqrt[3]{-\frac{\sqrt{5}}{2} - \frac{1}{2}} = -\sqrt[3]{\frac{\sqrt{5}}{2} + \frac{1}{2}}$

$\therefore$ Three fixed points

Note the polynomial has a double root at $x = 0 (= x_0)$ and single roots at $x = x_1 (>0)$, $x = x_2 (<0)$.

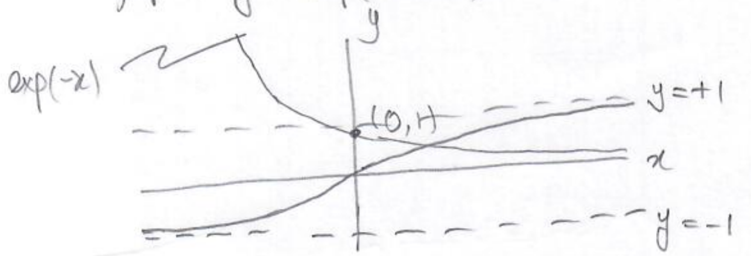
Sketch of f (locally "$-x^2$" at the origin). unseen.





(ii) $\dot{x} = \exp(-x) - \tanh(x)$

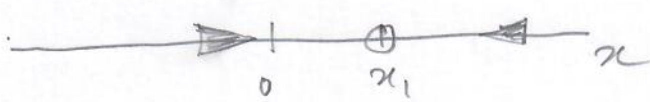
graphs of $\exp(-x)$, $\tanh(x)$ are:



$y = \tanh(x)$ monotonic incr

$y = e^{-x}$ " decr.

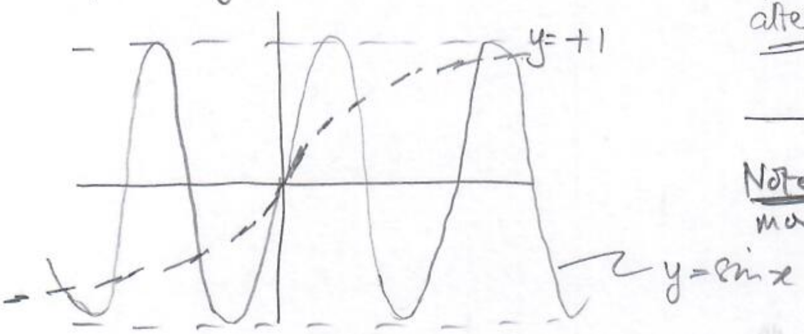
unique cross over $\therefore$ one fixed point $x = x_1$ for $x_1 > 0$.



seen similar

(iii) $\dot{x} = \sin x - \tanh(x)$

graphs of $\sin x$ & $\tanh(x)$



qualitative type of phase portrait
alternating stable and unstable points

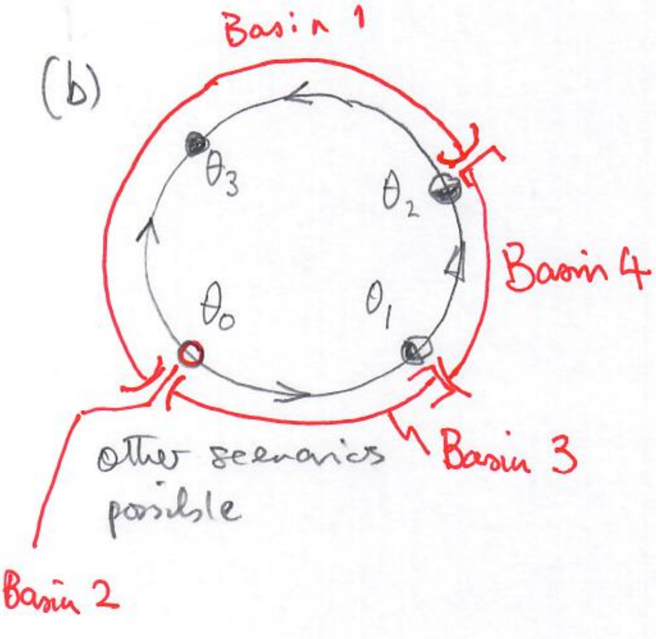
Note quantitatively pairs of points move closer together as $|x| \to \infty$.

both $\sin x$ and $\tanh x$ confined to $|y| \leq 1$. $\sin x$ acheives max and min of ± 1 at $x = \frac{\pi}{2} + 2n\pi$ (max) and $x = -\frac{\pi}{2} + 2n\pi$ (min) $n \in \mathbb{Z}$. whereas $|\tanh(x)| < 1 \ \forall x$ and $\tanh(x) \to \pm 1$ as $x \to \pm\infty$ respectively.

therefore
$y = \tanh(x)$ repeatedly intersects $y = \sin(x)$ beneath the max value $+1$ as x increases through +ve values and analogously for min value -1 as x decreases through −ve values.
At $x = 0$ $\sin x \approx x - \frac{x^3}{6}$, $\tanh(x) = x - \frac{x^3}{3}$ $\therefore \sin x - \tanh x \approx \frac{x^3}{6}$ $>0, x>0$; <0 $x<0$.

unseen

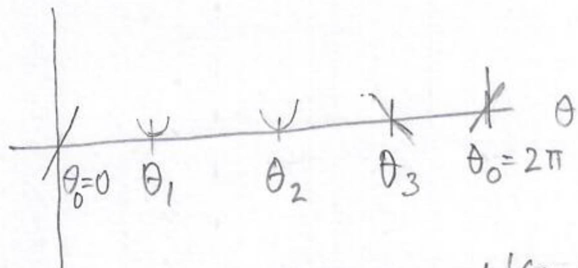
(b)



unseen, but partially similar examples

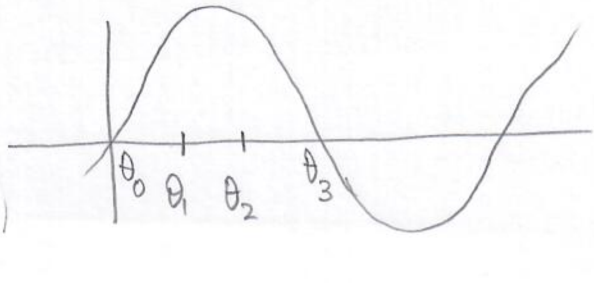
(c) Suppose the fixed points are at $\theta = \theta_1 = 0, \theta_2, \theta_3, \theta_4$

We need a function like



with $f'(0) > 0$, $f'(\theta_2) = 0$, $f'(\theta_3) = 0$ and $f'(\theta_4) < 0$

Consider $f(\theta) = \sin\theta$ and modify

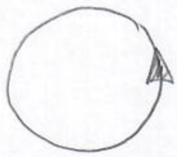


We need to modify $y = \sin\theta$ to provide zeroes at $\theta=\theta_1$, $\theta=\theta_2$ while preserving $f(\theta)\geq 0$ for $\theta\in[\theta_0,\theta_3]$ & $f(\theta)<0$ for $\theta\in[\theta_3,\theta_0+2\pi]$

The function $1-\sin(\theta-\theta_1)$ is >0 for $\theta\neq\theta_1$ & $=0$ for $\theta=\theta_1$

so the function $f(\theta)=\sin\theta\,(1-\sin(\theta-\theta_1))(1-\sin(\theta-\theta_2))$ provides the appropriate phase portrait.

(d) The phase portrait of $\dot\theta=f^2(\theta)$ clearly has the following form: either $f(\theta)\neq 0$ $\forall\ \theta\in[0,2\pi]$ which implies no fixed points and therefore a phase portrait of unrestricted rotation i.e.



or

the function $f(\theta)$ has zeroes, i.e. $\dot\theta=f^2(\theta)$ has fixed points $\theta=\theta_1,\theta_2,\ldots$ and between consecutive fixed points $f^2(\theta)>0$ so we have a sequence of saddle-nodes with counterclockwise flow between.

A degenerate case is $f(\theta)\equiv 0$ which provides a circle of fixed points

unseen

Question 2

(a) $\dot{x} = x(1+rx+x^2)$, $x \in \mathbb{R}$, $r \in \mathbb{R}$, parameter

(1) Fixed points :- (i) $x = 0$ $\forall r \in \mathbb{R}$ and

(ii) $1+rx+x^2=0$ $\forall r \in \mathbb{R}$.

Technique seen in lectures and examples

In the second case $x = \dfrac{-r \pm \sqrt{r^2-4}}{2}$

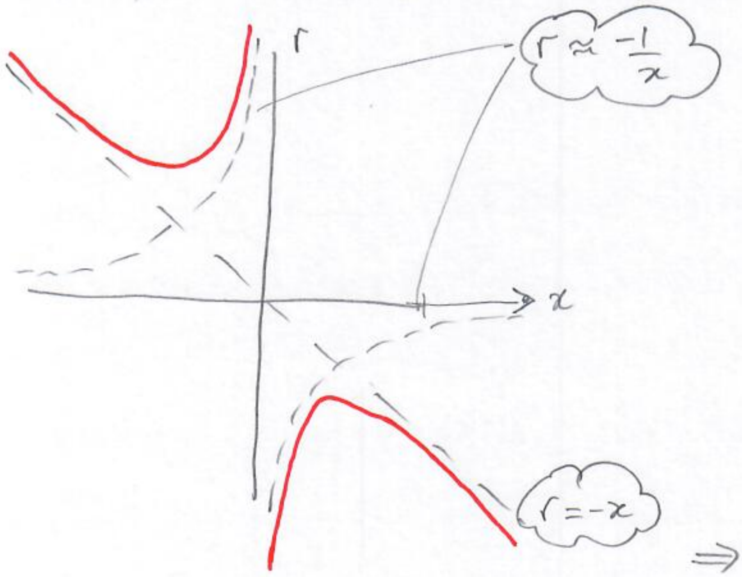
and so real roots occur only for $r^2-4>0$.

i.e. $r \geqslant 2$ or $r \leqslant -2$.

Solving $1+rx+x^2=0$ for r gives $r = -\dfrac{(1+x^2)}{x}$

and for $|x|$ close to zero we $r \approx -\dfrac{1}{x}$

and for $|x| \to \infty$ $r \approx -x$



Sketch suggests a min. for $r<0$ and max for $r>0$. of

$$r = -\frac{(1+x^2)}{x}$$

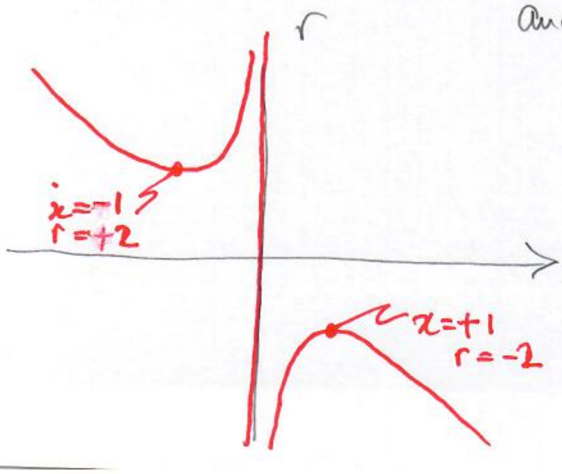
$$\frac{dr}{dx} = +\frac{1}{x^2} - 1 \Rightarrow x = \pm 1$$

$\Rightarrow$ $x=+1$, $r=-2$

$\Rightarrow$ $x=-1$, $r=2$

Note $\dfrac{d^2r}{dx^2} = \dfrac{-2}{x^3}$, and so $\dfrac{d^2r}{dx^2} > 0$ for $x=-1$ (Min)

and $\dfrac{d^2r}{dx^2} < 0$ for $x=+1$ (max)



fixed point set

Locate bifurcation points : $\dot{x} = f(x,r) = x(1 + rx + x^2)$ JAN21.5

$x = 0, \quad 1 + rx + x^2 = 0.$

$f'(x) = 1 + 2rx + 3x^2$

Checking the transitional points for umber of fixed points

$x=1,\ r=-2$: $f(1,-2) = 1(1+(-2)+1) = 0$

$\qquad f'(1,-2) = 1 - 4 + 3 = 0$

$x=-1,\ r=2 \quad f(-1,2) = 0$

$\qquad f'(-1,2) = 0$

Potential bifurcation points

(i) Local coordinates at $x=1,\ r=-2$, i.e. $y = x-1,\ \mu = r+2$

$$\dot{y} = (y+1)(1 + (\mu-2)(y+1) + (y+1)^2)$$
$$= (y+1)(1 + \mu y + \mu - 2y - 2 + y^2 + 2y + 1)$$
$$= (y+1)(\mu y + \mu + y^2) = \mu + \mu y + y^3 + y^2 + \mu y^2 + \mu y$$
$$= \mu + 2\mu y + (\mu+1)y^2 + y^3$$

In notation from notes $f(x,r) = a\mu + bx^2 + cx\mu + \dots$ etc.

$a \neq 0,\ b \neq 0 \ \therefore$ saddle node bifurcation (subcritical)

and fixed points in location of $(x,r) = (1,-2)$

occure for $\mu < 0$, i.e. $r < 2$.

$r=-2$:

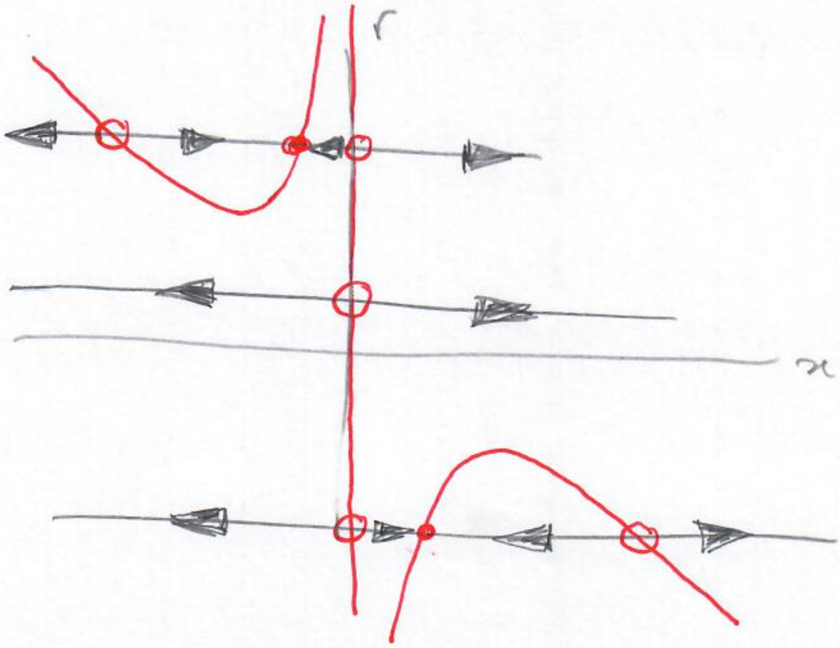
$r<-2$

(ii) Local coordinates at $x=-1,\ r=2 \quad y = x+1,\ \mu = r-2$

$$\dot{y} = (y-1)(1 + (\mu+2)(y-1) + (y-1)^2)$$
$$= \mu - (1-\mu)y^2 - 2\mu y + y^3 \qquad \text{coefficients } a, b \neq 0.$$

saddle-node – supercritical

(iii)



(b) $\dot{x} = x(1 + rx - x^2)$

Fixed points : $x = 0 \quad \forall\, r \in \mathbb{R}$ by inspection

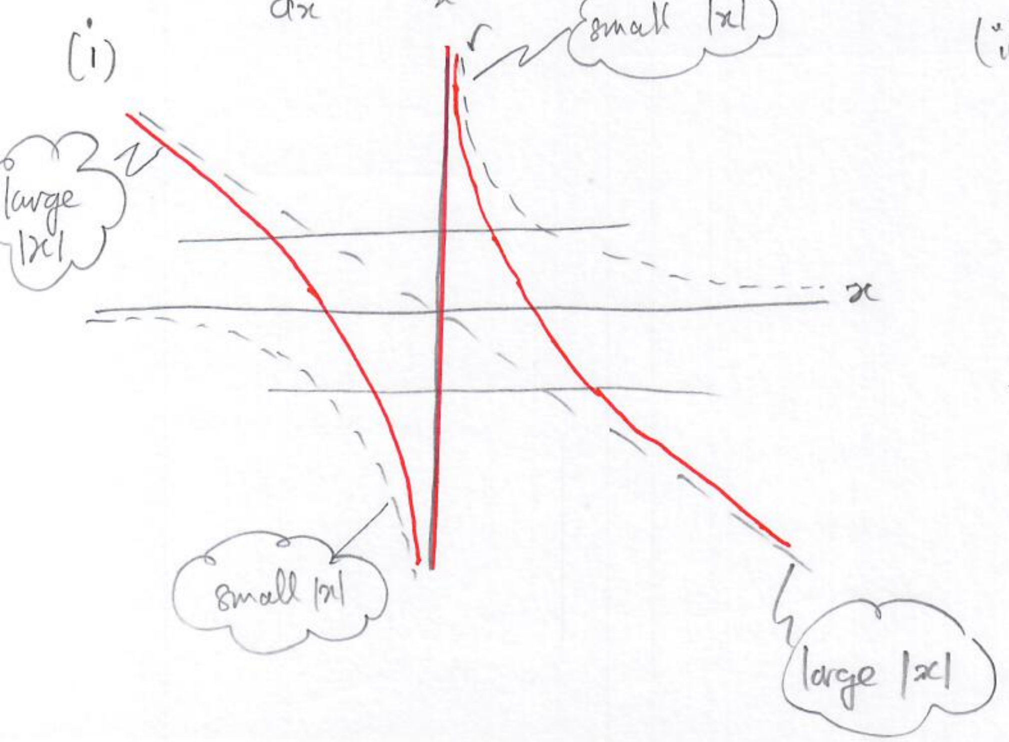
: $1 + rx + x^2 = 0 \iff r = -\frac{(x^2 - 1^2)}{x}$ for $x \neq 0$.

So cf. (a) $r \approx +\frac{1}{x}$ for $|x|$ small but

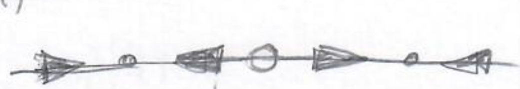
$r \approx -x$ for $|x|$ large.

Also $\frac{dr}{dx} = -\frac{1}{x^2} - 1 < 0 \Rightarrow$ no max/min.

(i)



(ii)



There is only one qualitative type of phase portrait.

There are no bifurcations.

not seen.
before.

Question 3

$\dot{x} = x(1-2y) \quad \dot{y} = -y(1-x) \quad \left(= \underline{f}(x,y)\right)$

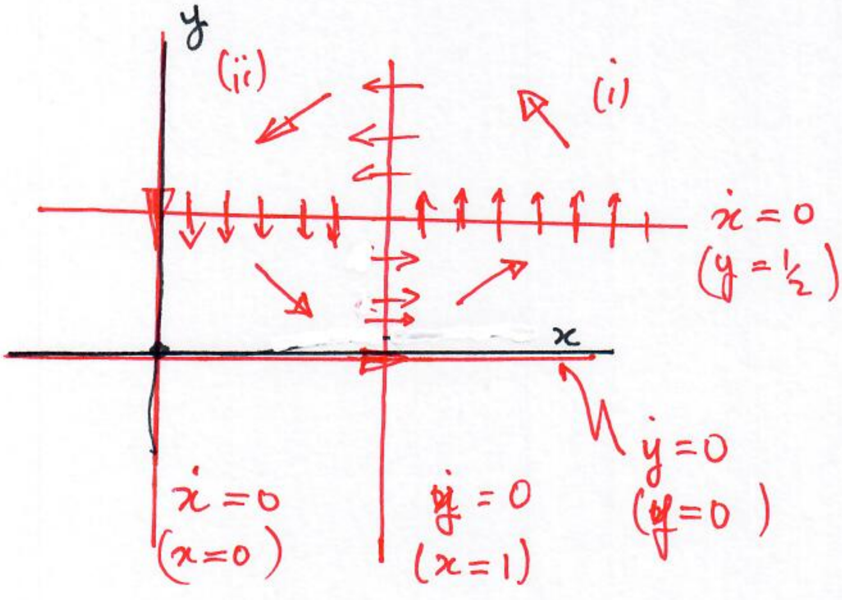
Plotting in the plane with conserved quantities

(1) fixed points $\quad (x,y) = (0,0) \quad , \quad (x,y) = (1, \tfrac{1}{2})$

Jacobians $\quad D\underline{f}(x,y) = \begin{bmatrix} 1-2y & -2x \\ y & -(1-x) \end{bmatrix}$

$Df(\underline{0}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$ eigenvalues: ± 1, eigenvectors: $(1,0)$, $(0,1)$ $\rightarrow$ saddle point (HGT applicable)

$Df(1, \tfrac{1}{2}) = \begin{bmatrix} 0 & -2 \\ \tfrac{1}{2} & 0 \end{bmatrix}$ eigenvalues: $\pm i$, eigenvectors: non-real $\rightarrow$ centre (HGT not applicable)



Quadrant behaviour in $x \geq 0, y \geq 0$.

(i) $x > 1$; $y > \tfrac{1}{2}$
$\dot{x} < 0 \quad \dot{y} > 0$

(ii) $x < 1$; $y > \tfrac{1}{2}$
$\dot{x} < 0 \quad \dot{y} < 0$

(iii) $x < 1$; $y < \tfrac{1}{2}$
$\dot{x} > 0$, $\dot{y} < 0$

(iv) $x > 1$, $y < \tfrac{1}{2}$
$\dot{x} > 0$, $\dot{y} > 0$

Suggest anti-clockwise rotational behaviour & saddle behaviour for x, y axes.

(ii) eliminating dt gives

$$\frac{dx}{x(1-2y)} = \frac{dy}{-y(1-x)} \Rightarrow \frac{1-x}{x}\,dx + dy\,\frac{(1-2y)}{y} = 0$$

i.e. integrating $\ln(x) - x + \ln(y) - 2y = \text{constant}$

is an integral solution.

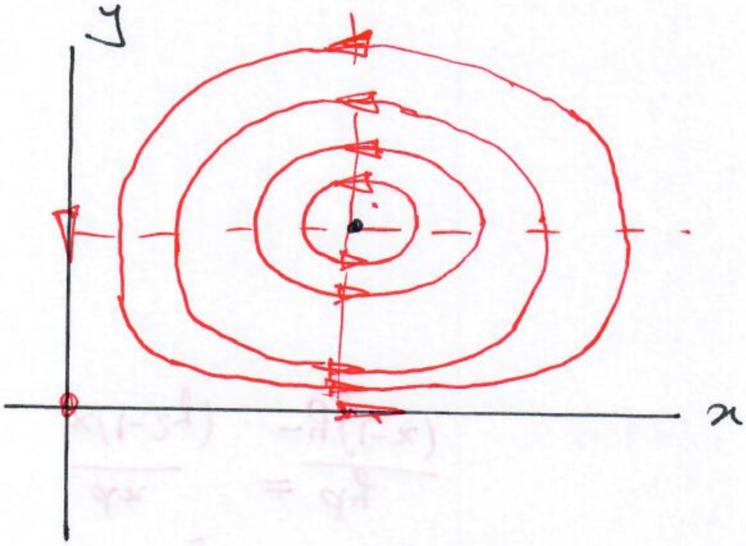
Let $f(x) = \ln(x) - x$ & $g(y) = \ln(y) - 2y$

$$f'(x) = \frac{1}{x} - 1 \Rightarrow x = 1 \text{ for } f' = 0$$

$$f''(x) = -\frac{1}{x^2} \quad \therefore \text{ max at } x = 1$$

Similarly $g(y)$ has a max at $y = \frac{1}{2}$.

(iii) $z = f(x) + g(y)$ surface has a maximum at $x = 1, y = \frac{1}{2}$ (the fixed point) and $z = \text{constant}$ gives conserved curves, i.e. closed curves. $\therefore$ system gives non-linear centre at the fixed point $x = 1, y = 1/2$.



(b) $L = (1-3x^2-y^2)^2$. Let $E = 1-3x^2-y^2$

Something simpler seen in examples.

$$\frac{dL}{dt} = 2E\frac{dE}{dt} = 2E\frac{\partial E}{\partial x}\cdot\frac{dx}{dt} + 2E\frac{\partial E}{\partial y}\frac{dy}{dt}$$

$$= 2E\Big[-6x\big(xE - y(1+x)\big) + (-2y)\big(yE + 3x(1+x)\big)\Big]$$

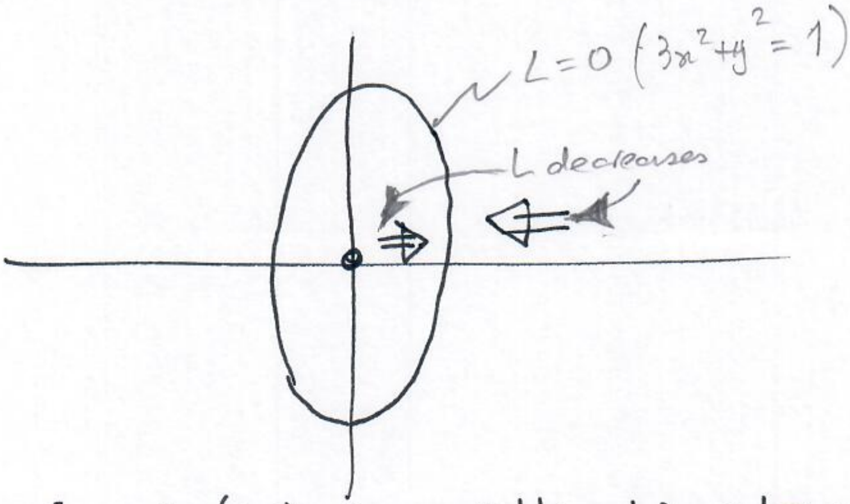
$$= 2E\left(-6x^2E - 2y^2E\right) = -2E^2(6x^2+2y^2)$$

$$= -4E^2(3x^2+y^2)$$

$\therefore \dfrac{dL}{dt} < 0$ for $3x^2+y^2 \neq 0$, $E \neq 0$, otherwise $\dfrac{dL}{dt} = 0$.

L decreases with t and has a minimum of $L=0$

at $1-3x^2-y^2=0$ i.e. $3x^2+y^2=1$,



Fixed point at (0,0) is unstable, linear type $\dot{x} = x-y$, $\dot{y} = y+3x$

Tr > 0 Det > 0, $Tr^2-4\,Det=-12$, ~~imag~~ complex eigenvalues unstable spiral $(Tr^2-4\,Det) = 4-16 = -12$

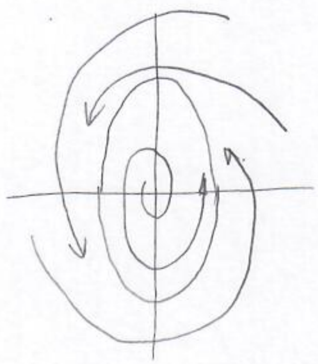
trajectories move out from the fixed point O and in from infinity towards the ellipse $3x^2+y^2=1$ (i.e. $L=0$).

So $3x^2+y^2=1$ is a stable limit cycle.

At $\underline{0}$, $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2} = \dfrac{(3x^2+y^2)(1+x)}{r^2}$ which is positive for (x,y) close to $\underline{0}$, so we have an anti-clockwise spiral.

On the curve $3x^2+y^2=1$, we have $|x| \le \frac{1}{\sqrt{2}} < 1$

and $\dot{\theta} = \dfrac{(3x^2+y^2)(1+x)}{r^2} = \dfrac{1}{r^2}(1+x) > 0$ $\therefore$ trajectory evolves anticlockwise. Qualitatively we get:



a stable limit cycle with anticlockwise flow near to the limit cycle.