COURSEWORK 3 QUESTIONS (STROGATZ)

ered even further before the pendulum will fail to make it over the top.

In more mathematical terms, we'll show in Section 8.5 that this hysteresis occurs because a *stable fixed point coexists with a stable periodic solution.* We have never seen anything like *this* before! For vector fields on the line, only fixed points can exist; for vector fields on the circle, both fixed points and periodic solutions can exist, *but not simultaneously.* Here we see just one example of the new kinds of phenomena that can occur in two-dimensional systems. It's time to take the plunge.

EXERCISES FOR CHAPTER 4

4.1 Examples and Definitions

4.1.1 For which real values of a does the equation $\dot{\theta} = \sin(a\theta)$ give a well-defined vector field on the circle?

For each of the following vector fields, find and classify all the fixed points, and sketch the phase portrait on the circle.

4.1.2 $\dot{\theta} = 1 + 2\cos\theta$

4.1.3 $\dot{\theta} = \sin 2\theta$

4.1.4 $\dot{\theta} = \sin^3\theta$

4.1.5 $\dot{\theta} = \sin\theta + \cos\theta$

4.1.6 $\dot{\theta} = 3 + \cos 2\theta$

4.1.7 $\dot{\theta} = \sin k\theta$ where k is a positive integer.

4.1.8 (Potentials for vector fields on the circle)

a) Consider the vector field on the circle given by $\dot{\theta} = \cos\theta$. Show that this system has a single-valued potential $V(\theta)$, i.e., for each point on the circle, there is a well-defined value of V such that $\dot{\theta} = -dV/d\theta$. (As usual, θ and $\theta + 2\pi k$ are to be regarded as the same point on the circle, for each integer k.)

b) Now consider $\dot{\theta} = 1$. Show that there is no single-valued potential $V(\theta)$ for this vector field on the circle.

c) What's the general rule? When does $\dot{\theta} = f(\theta)$ have a single-valued potential?

4.1.9 In Exercises 2.6.2 and 2.7.7, you were asked to give two analytical proofs that periodic solutions are impossible for vector fields on the line. Review these arguments and explain why they *don't* carry over to vector fields on the circle. Specifically, which parts of the argument fail?

4.2 Uniform Oscillator

4.2.1 (Church bells) The bells of two different churches are ringing. One bell rings every 3 seconds, and the other rings every 4 seconds. Assume that the bells have just rung at the same time. How long will it be until the next time they ring together? Answer the question in two ways: using common sense, and using the method of Example 4.2.1.

$\dot{\theta} = 1 + 2\cos\theta \; (= f(\theta))$

In[21]:=

Exer 4.2.2

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Plot[1 + 2 * Cos[x], {x, 0, 2 Pi}]
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Out[21]= Exer 4.2.2

fixed pts occur for $\dot{\theta} = 0$

i.e. $1 + 2\cos\theta = 0$

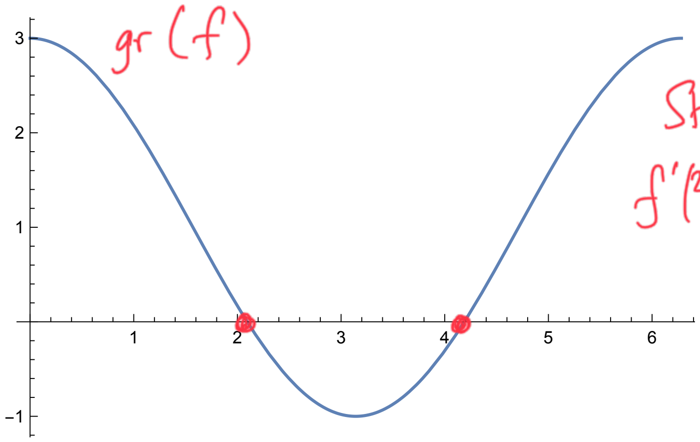
$\cos\theta = -\frac{1}{2} \Rightarrow \theta = 2\pi/3, \; 4\pi/3$

Stability: $f'(\theta) = -2\sin\theta$

$f'(2\pi/3) = -\sqrt{3}$, $f'(4\pi/3) = +\sqrt{3}$

(STABLE) (UNSTABLE)

Out[22]=



phase portrait

In[17]:=

Exer 4.2.3

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Plot[Sin[2 * x], {x, 0, 2 Pi}]
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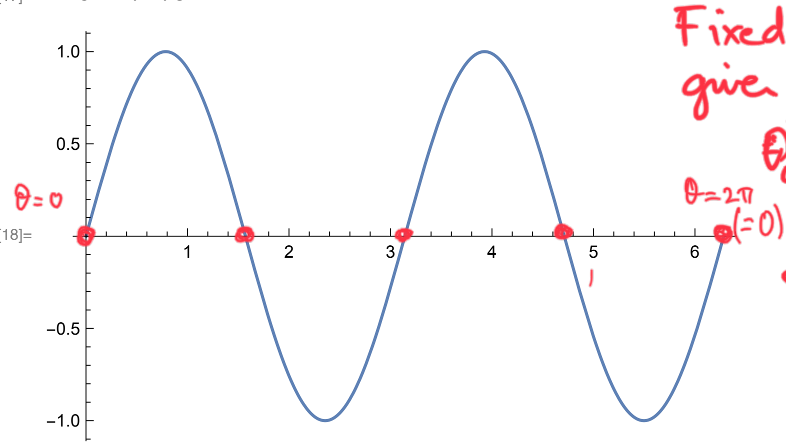
Out[17]= Exer 4.2.3

Out[18]=



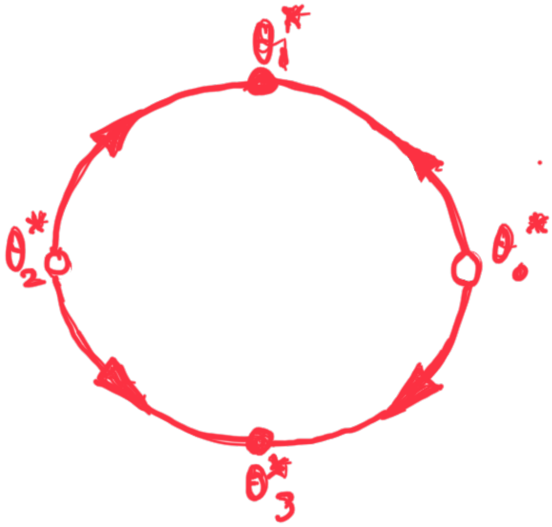
Fixed points of $\dot{\theta} = \sin 2\theta$ given by $\sin 2\theta = 0 \rightarrow$ 4 points

$\theta_0^* = 0, \; \theta_1^* = \frac{\pi}{2}, \; \theta_2^* = \pi, \; \theta_3^* = \frac{3\pi}{2}$

Stability check with $f(\theta) = \sin 2\theta$ & $f'(\theta) = 2\cos 2\theta$

θ_0^*, θ_2^* unstable

θ_1^*, θ_3^* stable



$\dot{\theta} = \sin\theta + \cos\theta.$

Note $\sin\theta + \cos\theta = \sqrt{2}\left(\sin\theta \frac{1}{\sqrt{2}} + \theta \frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\sin(\theta + \pi/4)$, hence curve shifted by $\pi/4$

fixed points, $\theta = 3\pi/4, 7\pi/4$

$\theta = \frac{3\pi}{4}$ stable fp

$\theta = \frac{7\pi}{4}$ unstable fp

In[15]:=

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Exer 4.1.5

Plot[Sin[x] + Cos[x], {x, 0, 2 Pi}]
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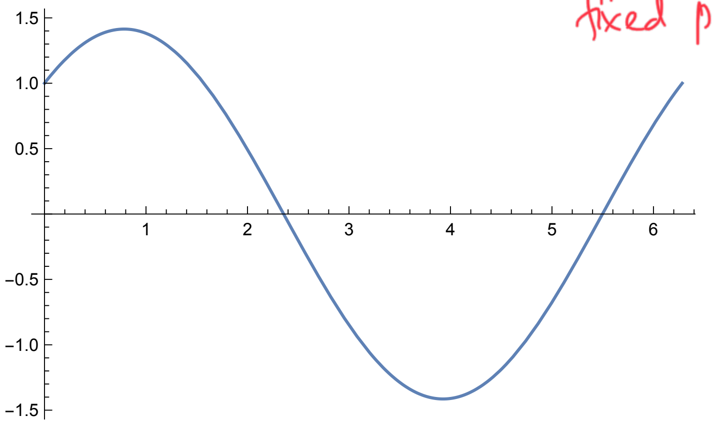
Out[15]= Exer 4.1.5

Out[16]=



In[12]:=

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Exer 4.1.7
k = 3;
Plot[Sin[k * x], {x, 0, 2 Pi}]
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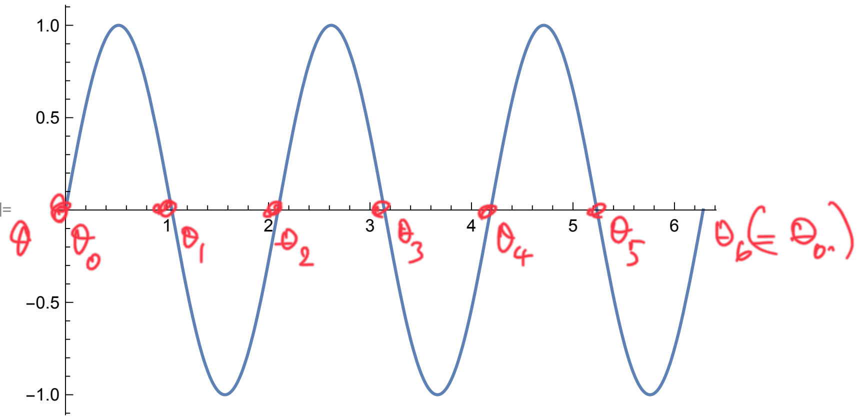
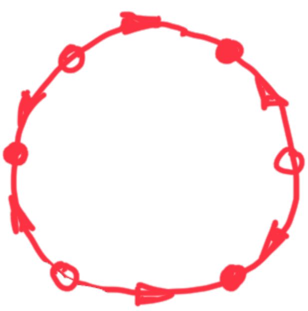
Out[12]= Exer 4.1.7

Out[14]=



Check with 4.1.3 & where k=2.

See k=3 above, there are 6 fixed points and the stability alternates $\underline{u, s}, \underline{u, s}, \underline{u, s}$



In general, 2k fixed points with alternating stability

$\underbrace{us}_{1}\ \underbrace{us}_{2}\ \ldots\ \underbrace{us}_{k}$

4.1.8

A potential is a function $V : S^1 \to \mathbb{R}$ such that if $\dot{\theta} = f(\theta)$, then $\frac{dV}{d\theta} = -f(\theta)$.

1. Consider $\dot{\theta} = \cos(\theta) \Rightarrow -\frac{dV}{d\theta} = \cos(\theta)$

$\Rightarrow V(\theta) = -\sin(\theta)$

Now V is single valued because on the circle a specific point is represented by $\theta + 2k\pi$, for any integer k. and $\sin(\theta) = \sin(\theta + 2k\pi)$ and so V is single-valued

2. Consider $\dot{\theta} = 1$, $-\frac{dV}{d\theta} = 1 \Rightarrow V = -\theta + C$.

Any specific point can be represented by $\theta + 2k\pi$, but this gives multiple values of V, i.e. $V(\theta + 2k\pi) = -(\theta + 2k\pi) \neq V(\theta)$ for $k \neq 0$.