Queen Mary
University of London

MTH6140: Linear Algebra II (Solutions)

Duration: 2 hours

Date and time: 31st April 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Question 1.

(a) uv and $a + u$ are invalid.

(b) (i) The list $(v_1, \ldots, v_n)$ is *linearly independent* if, for all $c_1, \ldots, c_n \in \mathbb{K}$, it is the case that $c_1v_1 + \cdots c_nv_n = 0$ implies $c_1 = \cdots = c_n = 0$; (ii) it is *spanning* if every vector $v \in V$ may be expressed in the form $v = c_1v_1 + \cdots c_nv_n$ for some $c_1, \ldots, c_n \in \mathbb{K}$; (iii) it is a *basis* if it is both linearly independent and spanning.

(c) Since the list is linearly dependent, there are scalars $c_1, \ldots, c_n$, not all zero, such that $c_1v_1 + \cdots + c_nv_n = 0$. Without loss of generality suppose $c_1 \neq 0$. Then we can write v_1 in terms of the other vectors:

$$v_1 = -(c_2/c_1)v_2 - \cdots - (c_n/c_1)v_n.$$

Since the list is spanning, any vector $u \in V$ can be written as a linear combination of $v_1, \ldots, v_n$. Substituting for v_1 we obtain an expression for u as a linear combination of $v_2, \ldots, v_n$. So removing v_1 leaves a list that is still spanning.

(d) Repeatedly remove vectors from the list, preserving the property of being spanning. Halt when the remaining list is linearly independent. The result is both linearly independent and spanning, and hence a basis.

(e) w_3 is a multiple of w_1 so we remove it. w_5 is half the sum of w_1 and w_4 so we remove it too. The remaining list (w_1, w_2, w_4) is clearly linearly independent and hence a basis. (Other possible outcomes are (w_2, w_4, w_5), (w_1, w_2, w_5), (w_2, w_3, w_4) and (w_2, w_3, w_5).)

Notes. Parts (c) and (d) are simplified versions of results proved in the course. Part (e) asks for the method of proof to be applied to a concrete example.

Question 2. (a) The matrices are, respectively,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) (i) $\det(A)$ is unchanged, (ii) the absolute value of $\det(A)$ is unchanged but its sign is reversed, and (iii) $\det(A)$ is multiplied by c.

(c) Let $R_1, \ldots, R_t$ be a sequence of elementary matrices (row operations) that reduce A to the identity (by multiplication on the left). Let $P = R_t \cdots R_2 R_1$. For any matrix X, it is the case that $\det(R_i X) = c_i \det(X)$ for some constant $c_i \neq 0$ depending on R_i; iterating, $\det(PX) = c \det(X)$ for some $c = c_1 \ldots c_t \neq 0$. Then $c \det(A) = \det(PA) = \det(I) = 1$ and $c \det(AB) = \det(PAB) = \det(B)$. It follows that $\det(AB) = c^{-1} \det(B) = \det(A) \det(B)$.

The identity also holds when either A or B is singular.

(d) A and B are similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

(e) With A and B as above,

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P) \\ &= \det(A) \det(P^{-1}) \det(P) = \det(A) \det(I) \\ &= \det(A). \end{aligned}$$

Notes. Parts (b)–(e) are bookwork and (a) a simple application of bookwork.

Question 3. (a) A function α from V to W is a linear map if $\alpha(u+v) = \alpha(u) + \alpha(v)$ and $\alpha(cu) = c\alpha(u)$, for all $u, v \in V$ and $c \in \mathbb{K}$.

(b) $\mathrm{Ker}(\alpha) = \{v \in V : \alpha(v) = 0\}$ and $\mathrm{Im}(\alpha) = \{\alpha(v) : v \in V\}$.

(c) First, all the vectors $\alpha(v_{k+1}), \ldots, \alpha(v_n)$ are clearly in $\mathrm{Im}(\alpha)$. We just need to show that any vector $v \in \mathrm{Im}(\alpha)$ can be expressed as a linear combination of $\alpha(v_{k+1}), \ldots, \alpha(v_n)$. Since $v \in \mathrm{Im}(\alpha)$ there is a vector $u \in V$ such that $v = \alpha(u)$. Also, since $v_1, \ldots, v_n$ is a basis of V, we can write u as $c_1v_1 + \cdots + c_nv_n$ for some scalars $c_1, \ldots, c_n$. Then

$$\begin{aligned} v = \alpha(u) &= \alpha(c_1v_1 + \cdots + c_nv_n) \\ &= c_1\alpha(v_1) + \cdots + c_n\alpha(v_n) \\ &= c_{k+1}\alpha(v_{k+1}) + \cdots + c_n\alpha(v_n), \end{aligned}$$

where we have used the fact that $v_1, \ldots, v_k \in \mathrm{Ker}(\alpha)$.

(d) $\dim(\mathrm{Ker}(\alpha)) + \dim(\mathrm{Im}(\alpha)) = \dim(V)$.

(e) Since $\mathrm{Ker}(\beta\alpha)) \supseteq \mathrm{Ker}(\alpha)$, we have

$$\dim(\mathrm{Ker}(\beta\alpha)) \geq \dim(\mathrm{Ker}(\alpha)) = \dim(V) - \dim(\mathrm{Im}(\alpha)) \geq \dim(V) - \dim(W).$$

Notes. Part (c) is a simplified version of something proved in the course. Part (e) is a thought part (though similar to some inequalities appearing in the exercise sheets).

Question 4. (a) The characteristic polynomial of a matrix A is defined to be $p_A(x) = \det(xI - A)$. The characteristic polynomial of a linear map α is the characteristic polynomial of any matrix representing α; it is well defined because the determinant function is invariant under the similarity relation. The minimal polynomial of α is the monic polynomial $m_\alpha(x)$ of smallest degree such that $m_\alpha(\alpha) = 0$. Such a polynomial exists by the Cayley-Hamilton Theorem.

(b)

$$p_A(x) = \begin{vmatrix} x-2 & 0 & 0 \\ -2 & x-5 & 2 \\ -3 & -6 & x+2 \end{vmatrix} = (x-2)\begin{vmatrix} x-5 & 2 \\ -6 & x+2 \end{vmatrix} = (x-2)^2(x-1).$$

The minimal polynomial divides $p_A(x)$ and has the same set of roots; thus either $m_A(x) = (x-2)(x-1)$ or $m_A(x) = (x-2)^2(x-1)$. Observe that

$$m_A(A) = (A-2I)(A-I) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & -2 \\ 3 & 6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \neq O,$$

eliminating the first possibility. So the only remaining possibility is $m_A(x) = (x-2)^2(x-1)$.

(c) A is not diagonalisable, since $m_A(x)$ has a repeated factor.

(d) The Jordan form for A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Parts (b)–(d) are fairly straightforward applications of definitions and facts from the course. In part (b), two marks each for the correct polynomials, and two and three marks respectively for the method.

Question 5. (a) The orthogonal complement of U is
$U^\perp = \{v \in V : v \cdot u = 0 \text{ for all } u \in U\}$.

(b) It is enough to show that $U^\perp$ is closed under vector addition and scalar multiplication. Suppose $v, v' \in U^\perp$ and $c \in \mathbb{R}$. Then, for all $u \in U$, we have that $(v + v') \cdot u = v \cdot u + v' \cdot u = 0 + 0 = 0$ and hence $v + v' \in U^\perp$. Similarly, $(cv) \cdot u = c(v \cdot u) = c0 = 0$ and hence $cv \in U^\perp$.

(c) $\dim(U) + \dim(U^\perp) = \dim(V)$.

(d) For $[a_1, a_2, a_3, a_4]^\top$ to be in $U^\perp$ it is sufficient that $a_1 + a_2 = 0$ and $a_3 + a_4 = 0$. We claim that the vectors $[1, -1, 0, 0]^\top$ and $[0, 0, 1, -1]^\top$ form a suitable basis for $U^\perp$. They are clearly independent and hence span a subspace of $\dim(U^\perp)$ of dimension 2. By part (c), we know that $\dim(U^\perp) = 2$, so that subspace is actually the whole of $U^\perp$.

(e) α is self-adjoint if $\alpha(v) \cdot w = v \cdot \alpha(w)$ for all $v, w \in V$.

(f) Observe that

$$\lambda(v \cdot w) = (\lambda v) \cdot w = \alpha(v) \cdot w = v \cdot \alpha(w) = v \cdot (\mu w) = \mu(v \cdot w).$$

Since $\lambda \neq \mu$ we must have $v \cdot w = 0$.

Notes. Part (b) is bookwork. Part (d) applies definitions and basic facts. Part (f) is a simplified version of something proved in the course. The rest is bookwork.

End of Paper.