Outline answers for MTH6140 Exam Jan 2023

Q1. (a) U is nonempty as $f = 0$ obeys the conditions. And at $a = 0, 2$, $(f + g)(a) = f(a) + g(a) = 0 + 0 = 0$, $(cf)(a) = cf(a) = c0 = 0$ if $f, g \in U$. So this meets the subspace test.
(b) $au_1 + bu_2 = a(x^2 - 2x) + b(x^3 - 2x^2) = bx^3 + (a - 2b)x^2 - 2ax$. So this $= 0$ implies $b, a = 0$ by coeffs of x^3 and x. Hence l.i.
(c) As U is 2-dimensional and u_1, u_2 l.i., they must be a basis and hence span.
(d) Over $\mathbb{F}_2$, $u_1 = x^2$ and $u_2 = x^3$ are l.i. as part of the standard basis of V. But now U is 3-dimensional so they cannot span (namely, $x \in U$ is not in the span of u_1, u_2).

Q2. (a) Bookwork to define $U + W$, $U \oplus W$.
(b) $\dim(U) = 1$ as U is a line, since intersection of two different planes (or solve for U as $z = -3x$ and $y = 5x$ for x free), while $\dim(W) = 2$ as a plane. $U \cap W = \{0\}$ as the line U through the origin is not in the plane W (or the joint solution of $0 = 3x - y + z = 5x$ requires $x = 0$).
(c) Yes, $\dim(U+W) = \dim(U)+\dim(W)-\dim(U\cap W) = 1+2-0 = 3 = \dim(V)$. So $U+W \subseteq V$ and has same dimension, hence $= V$.
(d) Yes, this is a direct sum since we have seen that $V = U + W$ and $U \cap W = \{0\}$.
(e) Yes, by Lectures, $V = U \oplus W$ implies projection $\pi : V \to V$ with $\mathrm{Image}(\pi) = U$ and $\ker(\pi) = W$. Fixing the standard basis of V, π corresponds to a matrix Π which obeys $\Pi^2 = \Pi$ and has $\mathrm{ColumnSpace}(\Pi) = U$.

Q3. (a) $S_2 = \{\mathrm{id}, (12)\}$, so $\det(A) = \mathrm{sign}(\mathrm{id})a_{11}a_{22} + \mathrm{sign}((12))a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}$ as $\mathrm{sign}((12)) = -1$ as for any transposition.
(b) By Lectures, det is linear in the first column, so this is $\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \begin{vmatrix} b & c \\ e & f \end{vmatrix}$.
(c) E.g. in sequence $c_1 \leftrightarrow c_2$, c_3-2c_1, c_4-3c_1, r_2-2r_1, $r_2/3$, c_3+c_1, c_4+2c_2 to get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.
(d) $\ker(\alpha)$ with respect to the basis means solution of $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}.[a, b, c, d]^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, solved by $d = -(b + 2c)/3$, $a = -(2b + c)/3$, for any b, c. So $\nu = 2$. Then $\rho = \dim(V) - \nu = 4 - 2 = 2$ by rank + nullity theorem.

Q4. (a) Bookwork defn of $m_A(x)$.
(b) First calculate $p_A(x) = \det(xI_3 - A) = (x - 3)(x^2 - 2x + 2) = (x - 3)(x - \lambda_+)(x - \lambda_-)$ where $\lambda_\pm = 1 \pm i$. As m_A has the same set of roots it must equal p_A.
(c) As m_A is the product of distinct linear factors, A diagonalisable by Lectures.
(d) Over $\mathbb{F}_2$, $p_A(x) = (x + 1)x^2$ from the first expression, as $2 = 0$ in the field. Now check $(A + I_2)A \neq 0$ hence $m_A = p_A$ again (and not $(x + 1)x$). But this has a repeated linear factor so A not diagonalisabe.

Q5. (a) $A = A^+ + A^-$ where $A^\pm = (A \pm A^t)/2$. Hence $q_A = q_{A^+} + \sum_{i,j}(\frac{a_{ij}-a_{ji}}{2})x_ix_j = q_{A^+}$ since the second term is 0 by cancelation with the same summand with $i \leftrightarrow j$. So the antisymmetric part of A does not affect q_A and we can suppose wlog that it is symmetric.
(b) $q_A = x^2 + 4xz + 2y^2 - z^2 = (x + 2z)^2 - 5z^2 + 2y^2 = u^2 + v^2 - w^2$ for $v = \sqrt{2}y$, $w = \sqrt{5}z$, $u = x+2z$ a linear change of variables. By Lectures this is equivalent to putting A into canonical form for congruence and we see $s = 2, t = 1$.
(c) $q(0, 0, 1) = -5 < 0$ so q is not positive definite, which is needed for its associated bilinear form to define an inner product.
(d) E.g. change A to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ then $q_A = (x + 2z)^2 + 2y^2 + z^2$ similarly to the above. But this is now $s = 3, t = 0$ so positive definite.