Outline answers for MTH6140 Exam Jan 2022

Q1. (a) We require $a_{ij} = -a_{ji}$, hence $a_{ii} = -a_{ii} = 0$ which for $\mathbb{R}$ means $a_{ii} = 0$. Hence $A = (a_{ij})$ is determined by a_{12}, a_{13}, a_{23} chosen freely and $a_{21} = -a_{12}$ etc. So V is 3-dimensional. (b) write $aA + bB + cC = 0$ where A, B, C are the three matrices shown. This gives $a + b = 0, a + c = 0, b + c = 0$ hence $b = -a, c = -a$ and $-2a = 0$. So over $\mathbb{R}$, all are zero, hence l.i. By dimensions, they then form a basis. (c) Over $\mathbb{F}_2$, the last equation does not imply $a = 0$ and a solution is $a = b = c = 1$, so l.d. (d) The space is *6* dimensional: as well as the three obvious matrices from (a) (with just one of a_{12}, a_{13}, a_{23} set to 1 and $a_{21} = a_{12}$ etc.) there are the three diagonal matrices with just one of a_{11}, a_{22}, a_{33} set to 1.

Q2. (a) Bookwork. Either define what means $V = U + W$ and ask for $U \cap W = \{0\}$ or say every element $v \in V$ can be *uniquely* written as $v = u + w$ with $u \in U$ and $w \in W$. (b) U=x-axis=$\langle(1, 0)\rangle$, W=y-axis=$\langle(0, 1)\rangle$. Every element $v = (x, y) \in V$ can be uniquely written $v = x(1, 0) + y(0, 1)$. Or, $U \cap W$ is the origin as the intersection of two lines so $= \{0\}$. (c) U, W as in (b) and $X = \langle(1, 1)\rangle$=the line at 45 degree slope. The intersections with U, W are again just the origin and every $v = (x, y) = x(1, 0) + y(0, 1) + 0(1, 1) \in U + W + X$ so $V \subseteq U + W + X$, and $U + W + X \subseteq V$ by definition, so $V = U + W + X$. (d) No, since if $V = (U \oplus W) \oplus X$ it would have dimension $\dim(U \oplus W) + \dim(X) = 2 + 1 = 3$ but V is 2-dimensional.

Q3. (a) The matrix C has the first and second rows swapped so $c_{1i} = 0$ unless $i = 2$, $c_{2i} = 0$ unless $i = 1$ and $c_{ji} = 0$ unless $i = j$ for $j = 3, \cdots, n$ (and has value 1 when not zero). Hence in the Leibniz formula $c_{1\pi(1)}c_{2\pi(2)} \cdots c_{n\pi(n)} = 0$ unless $\pi(1) = 2$, $\pi(2) = 1$ and $\pi(j) = j$ for $j > 2$, i.e. only $\pi = (12)$ contributes. In this case the product is 1 and $\mathrm{sign}(\pi) = -1$ so $\det(C) = -1$. (b) Write $v = av_1 + bv_2 + cv_3$ then $\alpha(v) = a\alpha(v_1) + b\alpha(v_2) + c\alpha(v_3) = (2a+b+c)w_1 + (a-b+c)w_2$. To be zero needs $2a + b + c = 0$, $b = a + c$, hence $a + 2b = 0$ giving $\ker(\alpha) = \langle 2v_1 - v_2 - 3v_3\rangle$. This is 1-dimensional so $\nu = 1$. (c) $\rho = \dim(V) - \nu = 3 - 1 = 2 = \dim(W)$ so by dimensions, $\mathrm{Im}(\alpha) = W$. (d) No. The matrix would need to be 2×3 matrix not a 3×2 so this is not possible. Or $\rho = \mathrm{rank}(A) = 2$ with respect to any basis but the matrix shown has rank 1. The latter was the intended answer as it had been intended to show a 2×3 matrix, but accepted either one.

Q4. (a) Bookwork from lectures that $p_A(x) = p_{PAP^{-1}}(x)$ hence this is true for the coefficients of each power of x (as these are a basis of the degree $\leq n$ polynomials in x). Hence $c(PAP^{-1}) = c(A)$ in particular. (b) By the Cayley-Hamilton theorem $p_A(A) = 0$ so $A^3 - \mathrm{Trace}(A)A^2 - \det(A)I_3 = 0$ as $c(A) = 0$. Hence $A^2(A - \mathrm{Trace}(A)I_3) = \det(A)I_3 = (A - \mathrm{Trace}(A)I_3)A^2$. As $\det(A) \neq 0$ we can divide through then this says A^2 has the inverse $(A - \mathrm{Trace}(A)I_3)/\det(A)$. (c) Expanding the determinant say on the top row, $p_A(x) = (x-1)(x(x+1)-2) - 1(-3(x+1)) = x^3 + 5$. (d) This *has* a single real root λ=minus the cube root of 5. The remaining roots must be complex so the $p_A = (x - \lambda I_3)q(x)$ where $q(x)$ has no real roots so does not factorise. m_A must divide p_A and $A - \lambda I_3 \neq 0$ so m_A is not this, hence must be all of p_A. Hence m_A is not a product of linear factors, hence A is not diagonalisable by a result in lectures.

Q5. (a) Bookwork, it means $B = P^TAP$ for an invertible $n \times n$ matrix P (b) Bookwork, $q_B(x') = \sum_{ij} x'_i b_{ij} x'_j = \sum_{ij} x_i a_{ij} x_j = q_A(x)$ for some linear change of variables $x_i \to x'_i$ (explicitly, $x_i = \sum_j P_{ij} x'_j$). (c) $q = (x+y)^2 + 2y^2 + z^2 = x'^2 + y'^2 + z'^2$ where $x' = x + y, y' = \sqrt{2}y, z' = z$. Either quote Sylveters that after this change of variables q is in the standard form with $s = 3 = \dim(V), t = 0$ hence positive definite, *or* argue directly that $q(x', y', z') \geq 0$ as a sum of squares and $= 0$ only if $x' = y' = z' = 0$, which is iff $x = y = z = 0$. Hence this meets the definition that $q(x, y, z)$ is positive definite. (d) We need $w_1.w_2 = acv_1.v_2 + bcv_2.v_2 = ac + 3bc = 0$ and $w_1.w_1 = a^2 v_1.v_1 + b^2 v_2.v_2 + 2ab v_1.v_2 = a^2 + 3b^2 + 2ab = 1$ and $w_2.w_2 = c^2 v_1.v_1 = 3c^2 = 1$. The last gives $c = 1/\sqrt{3}$ (say) then the first gives $a = -3b$, then the second gives $b = 1/\sqrt{6}$. Hence $w_1 = -\frac{3}{\sqrt{6}}v_1 + \frac{1}{\sqrt{6}}v_2$, $w_2 = \frac{1}{\sqrt{3}}v_2$, $w_3 = v_3$. (Also, $w_3.w_3 = v_3.v_3 = 1$ and $w_3.w_1 = w_3.w_2 = 0$.)