Outline answers for MTH6140 Exam Jan 2021

Q1. (a) Write $av_1 + bv_2 + cv_3 + dv_4 = 0$ and match the coefficients of each power x^i. This gives four equations $a + b + 2c + 6d = 0, a + 2b + 6c = 0, a + 3b = 0, a = 0$. Over $\mathbb{R}$ and $\mathbb{F}_5$, this gives $a = b = c = d = 0$ so l.i. Over $\mathbb{F}_3$, $v_4 = 0$ so the four vectors are not l.i.

(b) Yes over $\mathbb{R}$ and $\mathbb{F}_5$ as they are four l.i. vectors and the space has dimension 4 so they form a basis and hence span. No over $\mathbb{F}_3$ as only three nonzero vectors and need at least four to span.

Q2. (a) (i) adding a multiple c of a row j to a row i can be undone by adding a multiple $-c$ of row j to row i. (ii) Scaling row i by c can be undone by scaling by c^{-1}. (iii) Swapping rows i, j is its own inverse. ie doing it again undoes it.

(b) Do the row and col ops to get to canonical form and apply them separately to the relevant identity matrices to get P, Q. E.g. to get

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(c) Suppose for some fixed i that $a_{ki} = 0$ for any k. Then in the Leibniz formula $a_{1\pi(1)} \cdots a_{n\pi(n)} = 0$ for every π, since for every π there will be some k such that $\pi(k) = i$ so that $a_{k\pi(k)} = 0$.

(d) $\alpha(v_1) = w_1, \alpha(v_2) = w_2, \alpha(v_3) = 4w_1 + w_2$. Also $\rho(A) = 2$ from part (b) as D has rank 2.

Q3. (a) $\pi^2 = \pi$ and π linear. (b) If $v = c_1v_1 + c_2v_2$ then write each $v_i = u_i + w_i$ for $u_i \in U$ and $w_i \in W$. Hence $v = c_1(u_1 + w_1) + c_2(u_2 + w_2) = (c_1u_1 + c_2u_2) + (c_1w_1 + c_2w_2)$. The first term is $u \in U$ and the second is $w \in W$, say. So $\pi(v) = u = c_1u_1 + c_2u_2 = c_1\pi(v_1) + c_2\pi(v_2)$.

(c) (i) $\pi(v) = u \in U$ so $\mathrm{Im}(\pi) \subseteq U$. Conversely if $u \in U$ the $\pi(u) = u$ as $u = u + 0$ for $0 \in V$ is the unique decomposition of u. So $u \in \mathrm{Im}(\pi)$ so $U \subseteq \mathrm{Im}(\pi)$. (ii) $\pi(v) = 0$ says $v = 0 + w$ for $w \in W$ which happens iff $v \in W$. So $\ker\pi = W$.

(d) Define $\pi(A) = \frac{\mathrm{Tr}(A)}{n} I_n$ and check all the required properties hold.

Q4. (a) Bookwork (b) Bookwork, $p_B(x) = \det(xI_n - P^{-1}AP) = \det(P^{-1}(xI_n - A)P) = \det(P^{-1})\det(xI_n - A)\det(P) = p_A(x)$. (c) You should get $p_A(x) = (x - 1)^2(x - 2)$. So one option for m_A is $(x - 1)(x - 2)$ but check $(A - I_3)(A - 2I_2) \neq 0$ so then $m_A = p_A$. (d) m_A is not then the product of distinct linear factors so by a result in lectures A is not diagonalisable.

Q5. (a) $V = \mathbb{K} = \mathbb{F}_3$ and $q(v) = v^2$ for all $v \in \mathbb{F}_3$. (b) Bookwork, write $v = \sum_i x_iv_i$ to view v as a col vector $(x_1, \cdots, x_n)^T$. Then $q(v)$ becomes a function of the x_i where $a_{ij} = b(v_i, v_j)$ is defined from q.

(c) Basically bookwork, under a change of basis $v'_i = \sum_j P_{ij}v_j$ we have $v = \sum_i x'_iv'_i = \sum x_jv_j$ from which we find that $\sum_i x'_iP_{ij} = x_j$. Then $q(v) = \sum_{ij} a_{ij}x_ix_j = \sum_{ij}(PAP^T)_{lm}x'_lx'_m$, i.e in terms of x'_i the matrix $A = (a_{ij})$ has changed to $A' = PAP^T$. Sylvester's law says that by such congruence we can make A' diagonal with unique numbers s of 1's and t of -1's on the diagonal. So these s, t are associated to any quadratic form and found by a linear change of variables.

(d) $q(x, y, z) = -(x + y)^2 + (x + z)^2 + (y + z)^2$ using the hint. This $= x'^2 + y'^2 - z'^2$ for new variables $x' = x + y$, $y' = x + z$ and $z' = y + z$. So $s = 2$ and $t = 1$.