Queen Mary University of London

M. Sc. Examination by course unit 2012

ASTM002 The Galaxy

Duration: 3 hours

Date and time: 11 May 2012, 2:30pm .

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Examiner(s): W.J. Sutherland, J.P. Emerson

Useful information

In this paper π and e represent the conventional mathematical constants.
G is the gravitational constant, with $G = 6.67 \times 10^{-11}$ m^3 kg^{-1}s^{-2}.
c is the velocity of light, with $c = 3.00 \times 10^8$ m s^{-1}.
1 pc $=$ 3.09×10^{16} m.
1 astronomical unit (AU) $=$ 1.50×10^{11} m.
The mass of the Sun is $M_\odot = 2.0 \times 10^{30}$ kg.
The distance of the Sun from the Galactic Centre is $R_0 = 8.0$ kpc.

Poisson's equation states that $\nabla^2\Phi = 4\pi G\rho$ at any point in a gravitational field, where Φ is the gravitational potential, G is the constant of gravitation, and ρ is the mass density at that point.

The Laplacian of a scalar function Φ in a spherical coordinate system (r, θ, ϕ) is

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\Phi}{\partial r}\right) \;+\; \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) \;+\; \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2\Phi}{\partial\phi^2}\;.$$

The Laplacian of a scalar function Φ in a cylindrical coordinate system (R, ϕ, z) is

$$\nabla^2\Phi \;=\; \frac{1}{R}\,\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi}{\partial R}\right) \;+\; \frac{1}{R^2}\,\frac{\partial^2\Phi}{\partial\phi^2} \;+\; \frac{\partial^2\Phi}{\partial z^2}\;.$$

The Jeans equations in a steady-state (i.e. time-independent), spherically-symmetric galaxy give the following result

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n\,\langle v_r^2\rangle\Big) \;+\; \frac{n}{r}\left[\,2\langle v_r^2\rangle \;-\; \langle v_\theta^2\rangle \;-\; \langle v_\phi^2\rangle\right] \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\;,$$

in spherical coordinates, where n is the number density of stars at a distance r from the centre, v_r, v_θ and v_ϕ are the components of the velocity in the r, θ and ϕ directions, and $\Phi(r)$ is the gravitational potential.

In the absence of cosmological effects, the apparent magnitude m of an astronomical object in a photometric band is related to its absolute magnitude M in that band and its distance D from the observer by

$$m - M \;=\; 5\log_{10}(D/\mathrm{pc}) \;-\; 5 \;+\; A\;,$$

where A is the extinction in the band expressed in magnitudes.

[End of Useful Information]

Question 1 (a) Describe and compare the observed properties of Elliptical, Sa, and Sc type galaxies; including morphologies, colours, spectra, gas content and stellar populations. [8]

(b) A category of galaxies is found to have an observed surface brightness profile $I(R) = I_0 f(R/R_0)$ where f is some function, and I_0 and R_0 are constants for a given galaxy. Observations show that all the galaxies have the same I_0 and f, but different galaxies have different values of R_0.

Show that if the mass–to–light ratio has the same value in all these galaxies, then

$$L \propto v^4$$

where L is the total luminosity and v is a characteristic velocity. [3]

(c) Explain how the Tully-Fisher relation $L \propto v^4$ can be used to determine the distances of spiral galaxies. Observations show that one spiral galaxy (A) has $v_{rot} = 140\,\mathrm{km\,s^{-1}}$ and apparent magnitude $m_V = 14.0$, while a second spiral galaxy (B) has $v_{rot} = 220\,\mathrm{km\,s^{-1}}$ and apparent magnitude $m_V = 16.5$. Assuming galactic extinction is negligible, estimate the ratio of their distances. [5]

(d) Explain the difference between dissipational and dissipationless collapse in galaxy formation. Explain why gas is likely to settle to a rotating disc during galaxy formation. [4]

Question 2 An isolated system of N stars is bound by their own self-gravity. The ith star has a mass m_i, position vector $\mathbf{x}_i$ and velocity $\dot{\mathbf{x}}_i \equiv d\mathbf{x}_i/dt$ where t is time, and the origin is the centre of mass of the system. The total moment of inertia I of the system is defined as

$$I \equiv \sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \mathbf{x}_i \qquad .$$

(a) Show that

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_{i=1}^{N} m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i + \sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \ddot{\mathbf{x}}_i \qquad .$$

[4]

(b) Give an expression for T, the total kinetic energy of the system. [2]

(c) Give an expression for the gravitational force on star i due to star j in terms of vectors $\mathbf{x}_i$, $\mathbf{x}_j$. Hence write down an expression for the acceleration $\ddot{\mathbf{x}}_i$ of star i as a sum over $j \neq i$. [3]

(d) Hence, prove that

$$\sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \ddot{\mathbf{x}}_i = -\frac{1}{2} \sum_{i,j,(i \neq j)} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad ;$$

[5]

thus deduce the virial theorem,

$$2\langle T \rangle + \langle U \rangle = 0 \quad ,$$

where U is the total gravitational potential energy. [2]

(e) Describe the main assumptions and limitations in applying the virial theorem to real observational data. [4]

Question 3 (a) Explain the meaning of the terms *weak encounter* and *strong encounter* for two stars in a large stellar system approaching each other. [2]

(b) In a weak encounter between two stars each of mass m with relative velocity v, the change in the velocity of one star in the reference frame of the other is given by

$$\delta v = \frac{2Gm}{bv} \quad ,$$

where G is the constant of gravitation and b is the impact parameter.

A star moves through a spherical distribution of overall radius R containing N stars distributed uniformly in space. If the mean change in the square of the velocity is $\delta(v^2) = (\delta v)^2$ in a weak encounter, show that the changes in v^2 caused by all encounters with impact parameters in the range b to $b + \mathrm{d}b$ in a time t is

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3bvtN\mathrm{d}b}{2R^3}\right) \quad .$$

[5]

(c) Hence show that the total change in the square of the velocity in a time t caused by weak encounters with all impact parameters is

$$\Delta v^2(t) = 6\left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \quad ,$$

where b_{max} and b_{min} are the largest and smallest values of the impact parameter. [3]

(d) Define the crossing time $T_{\rm cross}$ and the relaxation time $T_{\rm relax}$.

For a stellar system of radius R containing N stars each of mass m, the typical velocity v is given by $v \approx \sqrt{GNm/R}$. From the above, derive an expression for the relaxation time, and show that for suitable choices of b_{min}, b_{max}, the ratio of the relaxation time to the crossing time is given approximately by

$$\frac{T_{\rm relax}}{T_{\rm cross}} \approx \frac{N}{6\ln N} \quad .$$

[7]

(e) When running a computer simulation, explain why two-body encounters may be neglected for a typical galaxy with $T_{\rm cross} \sim 10^8$ years, but not for a typical globular cluster with $T_{\rm cross} \sim 10^6$ years. [3]

Question 4 (a) The continuity equation for the distribution function f of stars in the six-parameter phase space $(x_1, x_2, x_3, v_1, v_2, v_3)$ of position $\mathbf{x}$ and velocity $\mathbf{v}$ states that

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i} \left(f \frac{\mathrm{d}x_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial v_i} \left(f \frac{\mathrm{d}v_i}{\mathrm{d}t} \right) \right) = 0 ,$$

where t is time.
Derive the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0$$

from the continuity equation, showing your working. [6]

(b) Derive the first of the Jeans equations,

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \langle v_i \rangle) = 0 ,$$

from the collisionless Boltzmann equation, where n is the number density of stars and $\langle v_i \rangle$ is the mean value of the v_i velocity component at a point. (Explain your working and assumptions). [8]

(c) One of the Jeans equations in a cylindrical coordinate system (R, θ, z) centred on the Galaxy, with $z = 0$ in the plane, gives

$$\frac{\partial (n \langle v_z \rangle)}{\partial t} + \frac{\partial (n \langle v_R v_z \rangle)}{\partial R} + \frac{\partial (n \langle v_z^2 \rangle)}{\partial z} + \frac{n \langle v_R v_z \rangle}{R} = - n \frac{\partial \Phi}{\partial z} ,$$

where n is the star number density, v_R and v_z are the velocity components in the R and z directions, $\Phi(R, z)$ is the Galactic gravitational potential and t is time. Assuming that the Galaxy is in a steady state, show that the surface mass density $\Sigma(z, R_0)$ within a distance z of the mid-plane of the Galactic disc at the solar radius R_0 is given by

$$\Sigma(z, R_0) = \frac{-1}{2\pi G n} \frac{\partial}{\partial z} \left(n \langle v_z^2 \rangle \right)$$

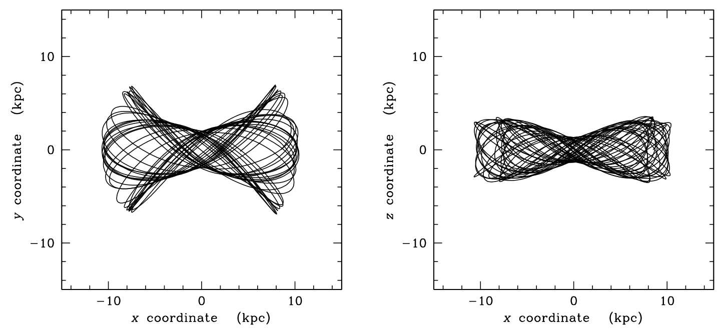
for stars lying towards the Galactic poles. Explain any other assumptions you make. [6]

Question 5 (a) Explain the difference between pressure support and rotational support for a galaxy. Which of these dominate for spiral and elliptical galaxies respectively ? [3]

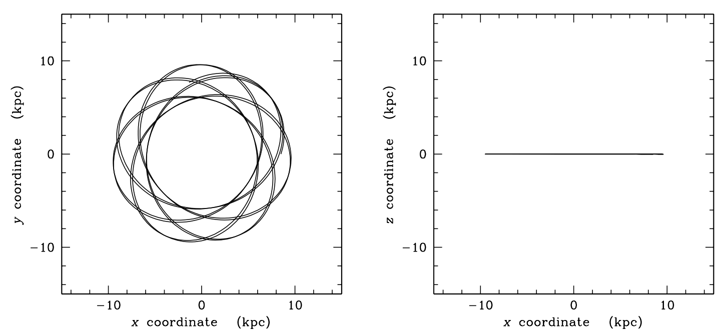
(b) Define the term *integral of motion* as applied to a stellar orbit. For an orbit in any time-independent axisymmetric potential, state *two* integrals of the motion. [3]

(c) The diagrams below show the orbit of a star in two gravitational potentials, shown projected in the $x - y$ and the $x - z$ planes.

Potential A:



Potential B:



What do you conclude about each of the potentials A and B: are they (i) spherical, (ii) flattened (oblate), or (iii) triaxial? Justify your answer on the basis of the character of the orbit. [4]

(d) A dark matter halo of a galaxy is sometimes modelled using a spherically-symmetric density distribution

$$\rho(r) = \frac{\rho_0}{1 + r^2/a^2} \quad ,$$

where $\rho(r)$ is the mass density at a radius r from the centre, and ρ_0 and a are positive constants. Show that for this distribution, the mass M interior to a radius r is

$$M(r) = 4\pi\rho_0 a^2 \left(r - a\tan^{-1}(r/a) \right) \ .$$ [5]

(You may assume the standard integral

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + \text{const} \qquad).$$

(e) Derive an expression for the circular velocity $v_{circ}(r)$ for the above mass distribution. What is the behaviour at radius $r \ll a$ and $r \gg a$ respectively ? [3]

(f) Discuss how realistic this density profile is for the Milky Way Galaxy, and how it may need to be modified at large distances. [2]

Question 6 (a) Explain the meanings of HI, HII, and H_2 referring to hydrogen in the interstellar medium of a galaxy. Give a description of the main observational methods for tracing each of these species, referring to the main emission processes and typical wavelengths involved. [8]

(b) Give a short discussion of dust in the interstellar medium, including its approximate composition, its importance to observations, and its importance to star and planet formation. [6]

(c) A star near the Galactic plane is observed to have apparent magnitudes in the blue and visual bands of $m_B = 15.48$, $m_V = 14.58$. Comparing the observed spectrum of the star to a library of local star spectra indicates that it has absolute magnitudes $M_B = 3.00$, $M_V = 2.50$. Assuming that the reddening ratio for interstellar dust is $A_V/E(B-V) = 3.0$, estimate (i) the reddening of the star, (ii) the V-band extinction in magnitudes, and (iii) the distance to the star. [6]

Question 7 (a) The symbols X, Y and Z denote the fractions by mass of Hydrogen, Helium and heavy elements (metals) respectively, with approximate values of $X \approx 0.71$, $Y \approx 0.27$, $Z \approx 0.02$. Which processes were responsible for creating (i) most of the helium, and (ii) most of the heavy elements ? [2]

(b) List any four assumptions behind the Simple Model of galactic chemical evolution. [4]

(c) In a region of the Galaxy, the total mass of stars is $M_{\rm stars}$, the total mass of interstellar gas is $M_{\rm gas}$, and the mass of heavy elements in the interstellar medium is $M_{\rm metals}$, while the metallicity of the gas is Z. The changes in these quantities in a small time interval are $\delta M_{\rm stars}$, $\delta M_{\rm gas}$, $\delta M_{\rm metals}$ and δZ respectively.
For the Simple Model of galactic chemical enrichment, derive the expression

$$\delta Z \;=\; \frac{\delta M_{\rm metals}}{M_{\rm gas}} \;-\; Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}}\;.$$

[4]

(d) If $\delta M_{\rm metals}$ and $\delta M_{\rm stars}$ above are related by
$\delta M_{\rm metals} \;=\; -\,Z\,\delta M_{\rm stars} \;+\; p\,\delta M_{\rm stars}$, where p is the yield of heavy elements, show that

$$\delta Z \;=\; -\,p\,\frac{\delta M_{gas}}{M_{gas}}\quad .$$

[3]

(e) Hence show that if $n(Z)$ is the number of stars with metallicity less than Z, and Z_1 is the present-day metallicity of the intergalactic medium, the Simple Model predicts that for long-lived stars we should observe

$$\frac{n(Z)}{n(Z_1)} \;=\; \frac{1 - e^{-Z/p}}{1 - e^{-Z_1/p}}\quad .$$

[5]

(f) How well does the above prediction match observations for G dwarfs in our galaxy ? [2]

Question 8 (a) General Relativity predicts that the bending angle α (assumed small) for a light ray passing a distance b from a compact object of mass M is given by

$$\alpha = \frac{4GM}{c^2 b} \ .$$

Using this, considering a gravitational lens system where the Earth, lens and source are exactly collinear, show that the physical Einstein ring radius is

$$r_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}D_L}{D_S}} \quad ,$$

where M is the lens mass, and D_L is the distance from Earth to the lens, D_S is the distance from Earth to the source, and D_{LS} is the distance between lens and source. [5]

(b) Assuming a lens mass $M = 0.01\,M_\odot$, an Earth–to–source distance $D_S = 50\,\mathrm{kpc}$, and a lens half-way between Earth and source, calculate the Einstein radius. Convert to astronomical units (see Useful Information). [4]

(c) The optical depth τ to microlensing is defined as the mean number of lenses within 1 r_E of the line of sight to a background source star. Show that the optical depth τ through a distribution of microlenses of mass M along a line of sight to a given source is given by

$$\tau \ = \ \frac{4\pi G}{c^2 D_S} \int_0^{D_S} D_L \, D_{LS} \, \rho(D_L) \, \mathrm{d}D_L \qquad ,$$

where $\rho(D_L)$ is the mean mass density of lenses at distance D_L. What is the approximate value of τ along a line of sight through the Galaxy ? [6]

(d) The optical depth to microlensing above is independent of compact-object masses for a given ρ; while actual microlensing experiments have ruled out the dark matter being mostly composed of compact objects between approximately $10^{-6}\,M_\odot$ and $10\,M_\odot$, but do not give useful limits outside this range. Give a short explanation of the physical reasons underlying these mass scales. (You may assume the typical lens velocity is $\sim 200\mathrm{km\,s^{-1}}$) [5]

End of Paper