MTH5123 Differential Equations
Exam 2017: Solution and Marking Scheme

All problems are modifications of problems that have been seen in the lectures and in the exercise classes

1. **a)** Find the general solution of the homogeneous ordinary differential equation (ODE) $y'' + 2y' - 15y = 0$. [**5 marks**]

   Solution: The characteristic equation is $\lambda^2 + 2\lambda - 15 = 0$ [**1**], which has two real roots: $\lambda_1 = -5$ and $\lambda_2 = 3$ [**2**]. The general solution of the homogeneous equation is thus given by $y_h(x) = c_1 e^{-5x} + c_2 e^{3x}$ [**2**].

   b) Find the general solution of the inhomogeneous ODE $y'' + 2y' - 15y = -4e^x$. [**11 marks**]

   Solution: Since the function e^x is not a solution of the homogeneous equation [**2**], we may use the *educated guess* method and look for a particular solution of the inhomogeneous equation in the form of $y_p(x) = d_0 e^x$ [**2**]. Substituting back into the inhomogeneous equation gives on the left-hand side $e^x d_0(1+2-15) = -12 d_0 e^x$ [**2**]. To match the right-hand side we have to choose $d_0 = 1/3$ so that $y_p(x) = e^x/3$ [**2**]. The general solution of the inhomogeneous equation is given by the sum $y_g(x) = y_h(x) + y_p(x)$ [**1**] yielding

   $$y_g(x) = c_1 e^{-5x} + c_2 e^{3x} + \frac{1}{3} e^x . \quad [\mathbf{2}]$$

   c) Find the general solution of the first order homogeneous linear ODE $y' = \tan(x)\, y$. [**5 marks**]

   Solution. The homogeneous ODE $y' = \tan(x)y$ is separable [**1**]. Following the standard procedure we introduce on the left-hand side $H(y) = \int \frac{dy}{y} = \ln|y|$ [**1**]. Solving $H(y) = u$ we find $y = \pm e^u = H^{-1}(u)$. On the right-hand side we have (see the formula sheet)

   $$\int \tan x \, dx = -\ln|\cos x| + C . \quad [\mathbf{1}]$$

   Hence the general solution of the homogeneous equation is given by

   $$y_h(x) = H^{-1}\left(-\ln|\cos x| + C\right) = \pm e^C \frac{1}{|\cos x|} = D \frac{1}{|\cos x|} , \quad [\mathbf{2}]$$

   where we denoted by $D = \pm e^C$ the constant of arbitrary sign.

d) Use the solution in c) to solve the initial value problem for the first order linear inhomogeneous ODE $y' = \tan(x)\, y + \sin x\,,\ -\pi/2 < x < \pi/2\,,\ y(0) = 1$ by the variation of parameters method. [**4 marks**]

Solution. According to the variation of parameters method we look for a solution of the inhomogeneous ODE in the form of

$$y = \frac{D(x)}{\cos x}\,.$$

Differentiating yields

$$y' = \frac{D'(x)}{\cos x} + D(x)\frac{\sin x}{\cos^2 x}\,. \quad [\mathbf{1}]$$

Substituting back into the equation $y' = \tan x\, y + \sin x$ we have

$$\frac{D'(x)}{\cos x} + D(x)\frac{\sin x}{\cos^2 x} = \frac{D(x)}{\cos x}\tan x + \sin x\,,$$

which implies

$$D'(x) = \cos x \sin x \quad \Rightarrow \quad D(x) = \frac{1}{2}\sin^2 x + C\,. \quad [\mathbf{1}]$$

This gives for the general solution of the inhomogeneous ODE

$$y_g(x) = \frac{1}{\cos x}\left(\frac{1}{2}\sin^2 x + C\right)\,. \quad [\mathbf{1}]$$

As $y(0) = C = 1$, we find the solution of the initial value problem

$$y_g(x) = \frac{1}{\cos x}\left(\sin^2 x + 1\right)\,. \quad [\mathbf{1}]$$

2. **a)** Find all functions $f(y)$ such that the following differential equation becomes exact:

$$x^2 + \frac{f(y)}{x} + \ln(xy)\frac{dy}{dx} = 0 \quad , \quad x > 0\,,\ y > 0\,. \qquad [\mathbf{5\ marks}]$$

Solution: Denoting $P(x,y) = x^2 + \frac{f(y)}{x}$, $Q(x,y) = \ln(xy)$ [**2**] we have $\frac{\partial P}{\partial y} = \frac{1}{x}\frac{df(y)}{dy}$, whereas $\frac{\partial Q}{\partial x} = \frac{1}{x}$ [**2**]. Hence the equation is exact only if $\frac{df(y)}{dy} = 1$ or equivalently $f(y) = y + C$ [**1**] with a real constant C.

b) Solve the equation in (a) in implicit form for a particular choice of $f(y)$ ensuring exactness such that $f(0) = 0$. **[11 marks]**

Solution: The condition $f(0) = 0 + C = 0$ implies $C = 0$ so that $f(y) = y$ [**2**]. Then the general solution should be looked for in implicit form as $F(x, y) = Const.$, where

$$F = \int P(x, y)\, dx = \int \left(x^2 + \frac{y}{x} \right) dx = \frac{x^3}{3} + y \ln x + g(y) \,. \quad [\mathbf{3}]$$

$g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = \ln x + g'(y)$ [**1**]. We conclude that $g'(y) = \ln y$ [**1**] so that (see formula sheet) $g(y) = \int \ln y dy = y \ln |y| - y$ [**2**]. Thus the solution in implicit form is

$$F(x, y) = \frac{x^3}{3} + y \ln x + y \ln y - y = Const.\,. \quad [\mathbf{2}]$$

c) Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \,,\ f(x, y) = \sqrt{25 + 4y^2} \,,\ y(1) = 0 \,.$$

Show that the Picard-Lindelöf Theorem guarantees the existence and uniqueness of the solution of the above problem in a rectangular domain $\mathcal{D} = (|x - a| \leq A\,, |y - b| \leq B)$ in the xy plane, and specify the parameters a and b. Find the possible range of values of the height B of the domain $\mathcal{D}$ given that the width A of the domain satisfies $A < 1/3$. **[9 marks]**

Solution: The right-hand side $f(x, y)$ is continuous everywhere, and its derivative $\frac{\partial f}{\partial y}$ satisfies $\left|\frac{\partial f}{\partial y}\right| = 4|y|/\sqrt{25 + 4y^2} < 2$ [**2**], hence is bounded. The initial conditions are $a = 1$ and $b = y(1) = 0$ [**1**]. Therefore, in the rectangular domain $\mathcal{D} = (|x - 1| \leq A, |y| \leq B)$ the solution of the ODE exists and is unique provided $A < B/M$ with $M = \max_{\mathcal{D}} \sqrt{25 + 4y^2}$ [**1**]. The function $f(x, y) = \sqrt{25 + 4y^2}$ on the right-hand side of the ODE grows with $|y|$. Thus, for a given B its maximum M is achieved for $|y| = B$ [**1**]. We then have $M = \sqrt{25 + 4B^2}$ [**1**], which implies that the width A should satisfy $A < B/M = B/\sqrt{25 + 4B^2}$ [**1**]. Requiring that the maximal value of the width $A = B/\sqrt{25 + 4B^2}$ fulfills $A < 1/3$ we obtain

$$B/\sqrt{25 + 4B^2} < 1/3 \quad [\mathbf{1}] \quad \Rightarrow \quad (3B)^2 < 25 + 4B^2 \text{ and } B < \sqrt{5} \quad [\mathbf{1}]\,.$$

Given $A < 1/3$, for these values of B existence and uniqueness of the solution are guaranteed.

3. Find the solution of the following boundary value problem (BVP) for the second order inhomogeneous ODE

$$\frac{1}{\cos x}\frac{d^2y}{dx^2} + \left(\frac{\sin x}{\cos^2 x}\right)\frac{dy}{dx} = f(x)\ ,\ y(0) = 0\ ,\ y\left(\frac{\pi}{4}\right) = 0$$

by using the Green's function method along the following lines:

a) Show that the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$. Use this fact to determine the general solution of the associated homogeneous ODE. [**4 marks**]

Solution: We have

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) = r(x)\frac{d^2y}{dx^2} + r'(x)\frac{dy}{dx}\ ,$$

which coincides with the original ODE for $r(x) = \frac{1}{\cos x}$. Therefore, the homogeneous ODE has the form

$$\frac{d}{dx}\left(\frac{1}{\cos x}\frac{dy}{dx}\right) = 0 \quad [\mathbf{2}]\ .$$

This can be integrated to find the general solution

$$\frac{1}{\cos x}\frac{dy}{dx} = C_1 \quad \Rightarrow \quad y(x) = C_1 \sin x + C_2 \quad [\mathbf{2}]$$

for real constants C_1 and C_2.

b) Formulate the left-end and right-end initial value problems corresponding to the above BVP. [**9 marks**]

Solution: The left-end boundary condition $y(0) = 0$ is imposed at $x_1 = 0$. By comparing it to the standard form $\alpha y'(x_1) + \beta y(x_1) = 0$ we conclude that $\alpha = 0, \beta = 1$ [**1**]. Then the left-end initial value problem for the function $y_L(x)$ is formulated as

$$y_L(x_1) = \alpha\ ,\ y_L'(x_1) = -\beta \quad \Rightarrow \quad y_L(0) = 0\ ,\ y_L'(0) = -1\ . \quad [\mathbf{2}]$$

Substituting the general solution of the homogeneous equation yields $C_2 = 0$, $C_1 = -1$ [**1**] so that

$$y_L(x) = -\sin x\ . \quad [\mathbf{1}]$$

Similarly, $x_2 = \frac{\pi}{4}$ and by comparing the right-end boundary condition $y\left(\frac{\pi}{4}\right) = 0$ to the standard form $\gamma y'(x_2) + \delta y(x_2) = 0$ we conclude that $\gamma = 0, \delta = 1$ [1]. Then the right-end initial value problem for the function $y_R(x)$ is formulated as

$$y_R(x_2) = \gamma \, , \; y_R'(x_2) = -\delta \quad \Rightarrow \quad y_R\left(\frac{\pi}{4}\right) = 0 \, , \; y_R'\left(\frac{\pi}{4}\right) = -1 \, , \quad [2]$$

which gives $\frac{1}{\sqrt{2}}C_1 + C_2 = 0$, $C_1 = -\sqrt{2}$ and thus $C_2 = 1$ so that

$$y_R(x) = -\sqrt{2}\sin x + 1 \, . \quad [1]$$

c) Use the solutions of these initial value problems to construct the Green's function $G(x,s)$ of the BVP. [**5 marks**]

Solution: Using $y_L(x), y_R(x)$ for the construction of the Green's function $G(x,s)$, first we calculate the Wronskian

$$W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

$$= -\sin s(-\sqrt{2}\cos s) - (-\sqrt{2}\sin s + 1)(-\cos s) = \cos s \, . \quad [1]$$

From the original ODE we have $a_2(s) = \frac{1}{\cos s}$ so that $a_2(s)W(s) = 1$, hence

$$A(s) = y_R(s) / \left(a_2(s)W(s)\right) = (-\sqrt{2}\sin s + 1),$$

$$B(s) = y_L(s) / \left(a_2(s)W(s)\right) = -\sin s \, . \quad [2]$$

The Green's function is then constructed as

$$G(x,s) = \begin{cases} A(s)y_L(x) \, , & 0 \le x \le s \\ B(s)y_R(x) \, , & s \le x \le \pi/4 \end{cases}$$

$$= \begin{cases} (\sqrt{2}\sin s - 1)\sin x \, , & 0 \le x \le s \\ (\sqrt{2}\sin x - 1)\sin s \, , & s \le x \le \pi/4 \end{cases} \, . \quad [2]$$

d) Write down the solution of the BVP in terms of $G(x,s)$ and $f(x)$. Use it to find the explicit form of the solution for $f(x) = 1$. [**7 marks**]

Solution: The solution of the boundary value problem is given by

$$y(x) = \int_0^{\pi/4} G(x,s)\, f(s)\, ds = \int_0^x G(x,s)\, f(s)\, ds + \int_x^{\pi/4} G(x,s)\, f(s)\, ds \, . \quad [2]$$

Substituting $G(x,s)$ and $f(x)=1$ we obtain

$$y(x)=\int_0^x(\sqrt{2}\sin x-1)\sin s\,ds+\int_x^{\frac{\pi}{4}}(\sqrt{2}\sin s-1)\sin x\,ds\,. \quad [\mathbf{1}]$$

After integration we get

$$y(x)=(\sqrt{2}\sin x-1)(-\cos s)|_0^x+\sin x(-\sqrt{2}\cos s-s)|_x^{\frac{\pi}{4}}\,, \quad [\mathbf{1}]$$

which can be written as

$$y(x)=(1-\cos x)(\sqrt{2}\sin x-1)+\sin x\left(-1-\frac{\pi}{4}+\sqrt{2}\cos x+x\right)\,. \quad [\mathbf{1}]$$

Simplification yields

$$y(x)=\sqrt{2}\sin x-1+\cos x-\sin x(1+\frac{\pi}{4})+x\sin x\,. \quad [\mathbf{2}]$$

4. Consider the system of two nonlinear first-order ODEs

$$\dot{x}=-4y-x^3\,,\ \dot{y}=3x-y^3\,. \qquad (1)$$

(a) Write down in matrix form the linear system obtained by linearization of the above equations around the point $x=y=0$. Then find the corresponding eigenvalues and eigenvectors. [**8 marks**]

Solution. Discarding the nonlinear terms we see that the matrix associated with the linearized system is given by $A=\begin{pmatrix}0 & -4\\ 3 & 0\end{pmatrix}$. [**1**] The characteristic equation is $\lambda^2+12=0$ [**1**], which has the two complex conjugate roots $\lambda_1=2i\sqrt{3}$, $\lambda_2=-2i\sqrt{3}$ [**1**]. Looking for the eigenvectors in the form $\mathbf{u}=\begin{pmatrix}p\\ q\end{pmatrix}$ we have for the eigenvector corresponding to λ_1

$$\begin{pmatrix}0 & -4\\ 3 & 0\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}=2i\sqrt{3}\begin{pmatrix}p\\ q\end{pmatrix}\,. \quad [\mathbf{2}]$$

This implies $-4q=2i\sqrt{3}p$, hence we can choose $p=1$ and $q=-i\frac{\sqrt{3}}{2}$ [**1**]. For the second eigenvalue we will have components obtained by complex conjugation of the above. The two eigenvectors are thus

$$\mathbf{u}_1=\begin{pmatrix}1\\ -i\frac{\sqrt{3}}{2}\end{pmatrix}\quad,\quad \mathbf{u}_2=\begin{pmatrix}1\\ i\frac{\sqrt{3}}{2}\end{pmatrix}\,. \quad [\mathbf{2}]$$

b) Determine the type of fixed point for the linear system. Is it stable? Is it asymptotically stable? Can one judge the stability of the nonlinear system by the linear approximation? [**4 marks**]

Solution. We see that the linear system is of *center type* [**1**], hence stable [**1**] but not asymptotically stable [**1**]. We cannot judge the stability of the zero equilibrium solution for the nonlinear system by its linear counterpart [**1**].

c) Write down the general solution of the linear system. [**2 marks**]

Solution. The general solution of the linear system is given by:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{2i\sqrt{3}t} \begin{pmatrix} 1 \\ -i\frac{\sqrt{3}}{2} \end{pmatrix} + C_2 e^{-2i\sqrt{3}t} \begin{pmatrix} 1 \\ i\frac{\sqrt{3}}{2} \end{pmatrix} . \quad [\mathbf{2}]$$

d) Find the solution of the linear system for the initial conditions $x(0) = 2$, $y(0) = 0$ in terms of real-valued functions. What is the shape of the corresponding trajectory in the phase plane? [**6 marks**]

Solution. From the general solution we have

$$x(t) = C_1 e^{2i\sqrt{3}t} + C_2 e^{-2i\sqrt{3}t} \quad \Rightarrow \quad x(0) = C_1 + C_2 = 2 \quad [\mathbf{1}]$$

$$y(t) = -i\frac{\sqrt{3}}{2}\left(C_1 e^{2i\sqrt{3}t} - C_2 e^{-2i\sqrt{3}t}\right) \quad \Rightarrow \quad y(0) = -i\frac{\sqrt{3}}{2}(C_1 - C_2) = 0\,, \quad [\mathbf{1}]$$

which gives $C_1 = C_2 = 1$ [**1**]. Hence the trajectory is determined by

$$x(t) = e^{2i\sqrt{3}t} + e^{-2i\sqrt{3}t} = 2\cos 2\sqrt{3}t \quad [\mathbf{1}]$$

$$y(t) = -i\frac{\sqrt{3}}{2}\left(e^{2i\sqrt{3}t} - e^{-2i\sqrt{3}t}\right) = \sqrt{3}\sin 2\sqrt{3}t \quad [\mathbf{1}]\ ,$$

which has the shape of an ellipse: $\frac{x^2}{4} + \frac{y^2}{3} = 1$ [**1**].

e) Demonstrate how to use the function $V(x,y) = 3x^2 + 4y^2$ to investigate the stability of the original nonlinear system (1). [**5 marks**]

Solution. It is $V(x,y) = 3x^2 + 4y^2 > 0$ for $(x,y) \neq (0,0)$ [**1**], and the orbital derivative is given by

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= 6x(-4y - x^3) + 8y\left(3x - y^3\right) = -6x^4 - 8y^4 < 0\,\forall\,(x,y) \neq (0,0)\,. \quad [\mathbf{3}]$$

Therefore $V(x,y)$ is a valid Lyapunov function ensuring the asymptotic stability of the solution of the nonlinear equation in the whole (x,y) plane [**1**].