Extra material: Connection between $2^{nd}$-order ODEs with constant coefficient and linear autonomous system of ODEs.

A $2^{nd}$-order ODE can be transformed into a system of first-order ODEs.

Example: Consider the ODE

$$\ddot{y} + \omega^2 y = 0 \qquad \text{with } \omega \in \mathbb{R} \quad \omega \neq 0$$

There are two possible methods to solve this ODE

(A) We can use the characteristic equation

$$M_2(\lambda) = \lambda^2 + \omega^2 = 0 \qquad \text{Roots} \quad \lambda^2 = -\omega^2 < 0$$

$$\lambda_1 = i\omega$$
$$\lambda_2 = -i\omega$$

The general solution of this ODE reads

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where $C_1 = C_2^*$ and $C_1$ is an arbitrary constant

$$y(t) = e^{\alpha t}\left(A \cos\beta t + B \sin\beta t\right)$$

$A, B \in \mathbb{R}$ arbitrary constants

$$\begin{cases} y(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ \dot{y}(t) = c_1 (i\omega) e^{i\omega t} + c_2(-i\omega) e^{-i\omega t} \end{cases}$$

(B) We can transform the $2^{nd}$-order ODE into a dynamical system

We set $$\begin{cases} y_1 = y(t) \\ y_2 = \dot{y}(t) = \dot{y}_1(t) \quad * \end{cases}$$

We have

$$\underset{\substack{\uparrow \\ y_2 = \dot{y}_1}}{\dot{y}_2 =} \ddot{y}_1 \overset{\substack{y = y_1 \\ \downarrow}}{=} \ddot{y} \underset{\substack{\uparrow \\ \ddot{y} = -\omega^2 y}}{=} -\omega^2 y \overset{\substack{y = y_1 \\ \downarrow}}{=} -\omega^2 y_1 \qquad \boxed{\dot{y}_2 = -\omega^2 y_1} \; **$$

Therefore our $2^{nd}$-order ODE is transformed into the dynamical system

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\omega^2 y_1 \end{cases} \Rightarrow \quad \dot{Y} = AY$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

Let us calculate the eigenvalues and eigenvectors of A.

## Eigenvalues

$$\det(A - \lambda \mathrm{Id}) = \det \begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix} = \lambda^2 + \omega^2 = 0$$

Roots $\lambda_1 = i\omega$

$\lambda_2 = -i\omega$

Complex conjugate eigenvalues $\alpha = \mathrm{Re}\,\lambda_1 = 0$

Phase portrait : CENTRE

## Eigenvectors

The eigenvector $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ associated to the eigenvalue $\lambda_1 = i\omega$ can be found by solving

$$A u_1 = \lambda_1 u_1$$

$$\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = i\omega \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}$$

$$\begin{cases} q_1 = i\omega p_1 \\ -\omega^2 p_1 = i\omega q_1 \end{cases}$$

$$(\cancel{i\omega})(i\omega) p_1 = \cancel{i\omega} q_1$$

$$-\omega^2 = (i\omega)(i\omega)$$

$p_1 = 1 \qquad q_1 = i\omega$

Since A is real and $\lambda_1, \lambda_2$ are complex conjugate

We can chose $u_2$

$$u_2 = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$$

i.e. $u_2$ is complex conjugate of $u_1$

In order to find the equations for $u_2$ we consider the

problem $\qquad A u_2 = \lambda_2 u_2 \qquad$ where $\quad u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = -i\omega \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

$$\begin{cases} q_2 = -i\omega p_2 \\ -\omega^2 p_2 = -i\omega q_2 \end{cases}$$

$p_2 = 1 \qquad q_2 = -i\omega$

$p_2 = (3-i) \quad q_2 = -i\omega(3-i)$

We chose $\quad u_2 = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$

The general solution of the system of ODES

If $u_1 = u_2^*$

$C_1 = C_2^*$

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{i\omega t} u_1 + C_2 e^{-i\omega t} u_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{i\omega t} \begin{pmatrix} 1 \\ i\omega \end{pmatrix} + C_2 e^{-i\omega t} \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$$

$$y_1 = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \qquad = y(t)$$

$$y_2 = C_1 (i\omega) e^{i\omega t} + C_2 (-i\omega) e^{-i\omega t} \qquad = \dot{y}(t)$$

The general solution obtained with method A and method B is the same

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$$y(0) = 1$$
$$\dot{y}(0) = 1 \qquad \text{I.C.}$$

$$y(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$\dot{y}(t) = C_1 (i\omega) e^{i\omega t} + C_2 (-i\omega) e^{-i\omega t}$$

$$C_1^* = C_2$$

$$1 = y(0) = C_1 + \underbrace{C_1^*}_{= C_2} = 2 \operatorname{Re} C_1$$

$$1 = \dot{y}(0) = i\omega \left( C_1 - C_1^* \right) = i\omega \left( 2i \operatorname{Im} C_1 \right) = -2\omega \operatorname{Im} C_1$$

$$\text{If } C_1 = \frac{\tilde{a}}{2} + i \frac{\tilde{b}}{2} \qquad \tilde{a}, \tilde{b} \in \mathbb{R}$$

$$1 = 2 \cdot \frac{\tilde{a}}{2} \qquad \Rightarrow \tilde{a} = 1$$

$$1 = -2\omega \frac{\tilde{b}}{2} \qquad \Rightarrow \tilde{b} = -\frac{1}{\omega}$$

$$y(t) = \overbrace{\left( \frac{1}{2} - \frac{1}{2\omega} i \right)}^{C_1} e^{i\omega t} + \overbrace{\left( \frac{1}{2} + \frac{1}{2\omega} i \right)}^{C_2 = C_1^*} e^{-i\omega t}$$