Exam structure (Jan / August)

1.	Multiple choice question	15 marks	Weeks 1-6
2.	Two questions		Weeks 1-6
3.	One question		Weeks 8-11 (week 10 excluded but make sure you can sketch phase portrait as in tutorial of week 10)

Summary of the module

Weeks 1-2 1^{st}-order ODEs.

Simplest (calculus like)

Separable ODEs

Reducible to separable
$$\begin{cases} y' = f(ax+by+c) \\ y' = f(y/x) \end{cases}$$

Exact ODEs

Variation of parameter method.

Week 3 IVP and the Picard-lindelöf theorem.

Weeks 4 - 5 2^{nd}-order ODEs

Linear 2^{nd}-order ODEs with constant coefficients

Euler ODEs

Inhomogeneous linear ODEs

- Variation of parameter method
- Educated guess method.

Week 6 The theorem of the Alternative

Week 8-11 Dynamical systems.

Week 8
- Equilibria
- linearisation around equilibrium point.

Week 9 Linear systems of ODEs - Algebra
Eigenvalues
Eigen vectors

General solution to a linear autonomous system of ODEs

Sketch phase portraits.

Week 10 Tutorial is examinable.

Week 11 Lyapunov stability, Asymptotic stability
Lyapunov functions, gradient flow.

Train in preparation for the exam.

Exam 2015 Question 2 a)

Find all functions $f(y)$ for which the following ODE is exact:

$$\frac{dy}{dx} = - \frac{x^3 + f(y)}{6xy^2 + 5y^4}$$

Rearranging the ODE:

$$(6xy^2 + 5y^4)\frac{dy}{dx} = -(x^3 + f(y))$$

$$\underbrace{(x^3 + f(y))}_{Q(x,y)} + \underbrace{(6xy^2 + 5y^4)}_{P(x,y)}\frac{dy}{dx} = 0$$

$$\begin{cases} Q(x,y) = \dfrac{\partial F}{\partial x} \\ P(x,y) = \dfrac{\partial F}{\partial y} \end{cases} \iff \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\boxed{\frac{\partial Q(x,y)}{\partial y} = \frac{\partial P(x,y)}{\partial x}}$$

$$\frac{\partial}{\partial y} Q(x,y) = \frac{\partial}{\partial y}\left(x^3 + f(y)\right) = f'(y)$$

$$\Rightarrow \quad f'(y) = 6y^2$$

$$\frac{\partial}{\partial x} P(x,y) = \frac{\partial}{\partial x}\left(6xy^2 + 5y^4\right) = 6y^2$$

$$f(y) = \int 6y^2\,dy + c' = 2y^3 + c'$$

$$\boxed{f(y) = 2y^3 + c}$$ where c is an arbitrary constant.

b) Suppose that $f(y)$ satisfies $f(1) = 0$. Solve the ODE in implicit form.

$$f(1) = 2y^3 + c\Big|_{y=1} = 2 + c' = 0 \quad \Rightarrow \quad c' = -2$$

$$f(y) = 2y^3 - 2$$

The ODE is exact and has solution $F(x,y) = c'$

where

$$\frac{\partial F}{\partial x} = Q(x,y) = x^3 + \underbrace{2y^3 - 2}_{f(y)}$$

$$\frac{\partial F}{\partial y} = P(x,y) = 6xy^2 + 5y^4$$

$$F = \int Q(x,y)\,dx + g(y) = \int (x^3 + 2y^3 - 2)\,dx + g(y)$$

$$F = \frac{1}{4}x^4 + x(2y^3 - 2) + g(y)$$

$$P(x,y) = 6xy^2 + 5y^4 = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{4}x^4 + x(2y^3-2) + g(y)\right]$$

$$6\cancel{x}y^2 + 5y^4 = 6\cancel{x}y^2 + g'(y)$$

$$g'(y) = 5y^4 \quad \Rightarrow \quad g(y) = \int 5y^4\,dy = y^5 + C$$

$$F = \frac{1}{4}x^4 + x\,2y^3 - 2x + y^5 + C$$

General solution in implicit form is $F(x,y) = C'$

$$\frac{1}{4}x^4 + 2xy^3 - 2x + y^5 = C'' \qquad C'' = C' - C$$

C'' is an arbitrary constant.

$\square$

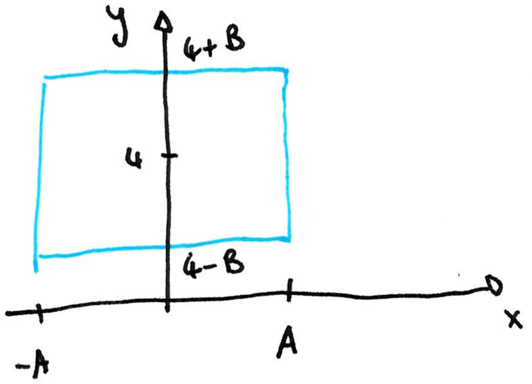
Question 2 Exam 2022-2023

(a) Check whether the IVP

$$y' = \frac{x}{y-4} = f(x,y) \qquad y(0) = 4 \qquad \text{I.C. } (0,4) = (a,b)$$

satisfies the ~~hyph~~ hypotheses of the Picard-Lindelöf theorem

Consider D: $\quad |x-a| \leq A$, $\quad |y-b| \leq B$ $\quad \Rightarrow \begin{cases} |x| \leq A \\ |y-4| \leq B \end{cases}$



① $f(x,y)$ is continuous in D.

$$f(x,y) = \frac{x}{y-4}$$ diverges for $y \to 4$

$$4 - B \leq y \leq 4 + B$$

There is NO $B>0$ such that $f(x,y)$ is continuous in D.

The hypotheses of the Picard-Lindelöf theorem are not satisfied