Lypunov function and gradient flow

Exercises in preparation of the exam

Mock quiz week 11

Which of the following functions $V(y_1, y_2)$ is a Lyapunov function for the zero solution of the dynamical system

$$\dot{y}_1 = -y_1 y_2^2$$

$$\dot{y}_2 = -2y_1^2 y_2 - 3y_2$$

We need to check whether $V(y_1, y_2)$ satisfy:

1) $V(0,0) = 0$

2) $V(y_1, y_2) > 0 \quad \forall (y_1, y_2) \neq (0,0)$

3) $D_f(V) \leq 0$ or 3') $D_f(V) < 0$ for $(y_1, y_2) \neq (0,0)$

a) $V(y_1, y_2) = -y_1 y_2$

- $V(0,0) = 0 \quad ✓$
- $V(y_1, y_2) = -y_1 y_2 \qquad y_2 = 0 \quad y_1 = \varepsilon \quad \times$

$$V(0, \varepsilon) = 0$$

b) $V(y_1, y_2) = \sin(y_1^2 + y_2^2)$

$V(0,0) = 0$

$V(y_1, y_2) = \sin(y_1^2 + y_2^2)$ $\quad (y_1, y_2) \neq (0,0)$

c) $V(y_1, y_2) = y_2^2$

$V(0,0) = 0$

$V(y_1, y_2) = y_2^2$ $\quad \times \quad y_2 = 0 \quad y_1 = \varepsilon \quad V(y_1, y_2) = 0$

d) $V(y_1, y_2) = \sinh(y_1^2 + y_2^2)$

- $V(0,0) = \sinh 0 = 0$
- $V(y_1, y_2) = \sinh(y_1^2 + y_2^2) > 0$ $\quad \forall (y_1, y_2) \neq (0,0)$

because $\sinh z > 0$ if $z > 0$

$$D_f(V) = \frac{\partial V}{\partial y_1} \cdot \dot{y}_1 + \frac{\partial V}{\partial y_2} \cdot \dot{y}_2 =$$

$$V = \sinh(y_1^2 + y_2^2)$$

$$\frac{\partial V}{\partial y_1} = \cosh(y_1^2 + y_2^2)\,(2y_1) \qquad \dot{y}_1 = -y_1 y_2^2$$

$$\frac{\partial V}{\partial y_2} = \cosh(y_1^2 + y_2^2)\,(2y_2) \qquad \dot{y}_2 = -2y_1^2 y_2 - 3y_2$$

$$D_f(V) = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 = \cosh(y_1^2+y_2^2)(2y_1)\left[-y_1 y_2^2\right] +$$

$$\cosh(y_1^2+y_2^2)\,(2y_2)\left[-2y_1^2 y_2 - 3y_2\right] =$$

$$= \cosh(y_1^2+y_2^2)\,2\left[-y_1^2y_2^2 - 2y_1^2y_2^2 - 3y_2^2\right] =$$

$$= \cosh(y_1^2+y_2^2)\,2\left[-3y_1^2y_2^2 - 3y_2^2\right] =$$

$$= -6\cosh(y_1^2+y_2^2)\left(y_1^2y_2^2 + y_2^2\right) < 0 \quad \forall (y_1,y_2) \neq (0,0)$$

This is a Lyapunov function, the zero solution is

ASYMPTOTICALLY STABLE.

Gradient flow exercise

Verify that the following dynamical system is a gradient flow and determine its potential function V. Draw conclusions on the stability of the zero solutions $(y_1, y_2) = (0,0)$

$$\begin{aligned} \dot{y}_1 &= -y_1 - y_1 y_2^2 = f_1(y_1, y_2) \\ \dot{y}_2 &= -2y_2 - y_1^2 y_2 = f_2(y_1, y_2) \end{aligned} \qquad (1)$$

We want to express the dynamical system as a gradient flow, finding V such that

$$\dot{y}_1 = f_1(y_1, y_2) = -\frac{\partial V}{\partial y_1}$$

$$\dot{y}_2 = f_2(y_1, y_2) = -\frac{\partial V}{\partial y_2}$$

You can check if the system is a gradient flow by considering

$$\frac{\partial^2 V}{\partial y_1 \partial y_2} = \frac{\partial^2 V}{\partial y_2 \partial y_1} \Rightarrow \frac{\partial f_1}{\partial y_2} = \frac{\partial f_2}{\partial y_1}$$

In order the check that (1) is a gradient flow.

LHS: $\frac{\partial f_1}{\partial y_2} = \frac{\partial}{\partial y_2}\left(-y_1 y_2^2\right) = -2y_1 y_2$ ✓

RHS: $$\frac{\partial f_2}{\partial y_1} = \frac{\partial}{\partial y_1}\left(-2y_2 - y_1^2 y_2\right) = -2y_1 y_2 \quad ✓$$

(1) is a gradient flow !

We need to find V such that

$$f_1(y_1,y_2) = -y_1 - y_1 y_2^2 = -\frac{\partial V}{\partial y_1} \qquad a)$$

$$f_2(y_1,y_2) = -2y_2 - y_1^2 y_2 = -\frac{\partial V}{\partial y_2} \qquad b)$$

Integrating a) we set

$$V = -\int f_1(y_1,y_2)\, dy_1 + g(y_2) = \int (y_1 + y_1 y_2^2)\, dy_1 + g(y_2) =$$

$$V = \frac{y_1^2}{2} + \frac{1}{2} y_1^2 y_2^2 + g(y_2)$$

Using b)

$$f_2(y_1,y_2) = -2y_2 - y_1^2 y_2 = -\frac{\partial V}{\partial y_2} = -\frac{\partial}{\partial y_2}\left(\frac{y_1^2}{2} + \frac{1}{2} y_1^2 y_2^2 + g(y_2)\right)$$

$$-2y_2 - \cancel{y_1^2 y_2} = -\left[\cancel{y_1^2 y_2} + g'(y_2)\right]$$

$$g'(y_2) = 2y_2$$

$$g(y_2) = \int 2y_2 \, dy_2 = y_2^2 + C$$

$$V = \frac{y_1^2}{2} + \frac{1}{2} y_1^2 y_2^2 + y_2^2 + C$$

- $V(0,0) = 0 \implies V(0,0) = C = 0 \qquad C = 0$

- $V(y_1, y_2) = \frac{y_1^2}{2} + \frac{1}{2} y_1^2 y_2^2 + y_2^2 > 0 \quad \forall (y_1, y_2) \neq (0,0)$

$V = \frac{y_1^2}{2} + \frac{1}{2} y_1^2 y_2^2 + y_2^2$ is the potential function of the gradient flow.

Is the zero solution Lypunov stable or also asymptotically stable?

$$D_f(V) = \frac{\partial V}{\partial y_1} \dot{y}_1 + \frac{\partial V}{\partial y_2} \dot{y}_2 = - \left[\left(\frac{\partial V}{\partial y_1} \right)^2 + \left(\frac{\partial V}{\partial y_2} \right)^2 \right] =$$

$$\dot{y}_1 = - \frac{\partial V}{\partial y_1}$$

$$\dot{y}_2 = - \frac{\partial V}{\partial y_2}$$

$$D_f(V) = - \left[(y_1 + y_1 y_2^2)^2 + (2y_2 + y_1^2 y_2)^2 \right] = - \left[y_1^2 (1 + y_2^2)^2 + y_2^2 (2 + y_1^2)^2 \right]$$

$$D_f(V) < 0 \qquad \forall\, (y_1, y_2) \neq (0,0)$$

It follows that the zero solution is ASYMPTOTICALLY STABLE.