Find the solution (implicit solution) to the following exact ODE

$$\frac{y}{x\ln|x|} + \left(\ln|\ln|x|| + y\right)\frac{dy}{dx} = 0$$

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

$$P(x,y) = \frac{y}{x\ln|x|} = \frac{\partial F}{\partial x} \qquad Q(x,y) = \ln|\ln|x|| + y = \frac{\partial F}{\partial y}$$

Implicit solution

$$F(x,y) = C$$

$$F(x,y) = \int P(x,y)\,dx + g(y) \;*\; = \int \frac{y}{x\ln|x|}\,dx + g(y)$$

$$= y\int \frac{1}{x\ln|x|}\,dx + g(y)$$

$$\int \frac{1}{x\ln|x|}\,dx = \int \frac{1}{z}\,dz \qquad z = \ln|x| \qquad dz = \frac{1}{x}\,dx$$

$$= \ln|z| = \ln|\ln|x|| = \int \frac{1}{x\ln|x|}\,dx$$

$$F(x,y) = y\ln|\ln|x|| + g(y)$$

Find the solutions of the following exact ODE

$$\frac{y}{x \ln|x|} + \left(\ln|\ln|x|| + y\right)\frac{dy}{dx} = 0$$

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

$$P(x,y) = \frac{y}{x\ln|x|} \qquad Q(x,y) = \ln|\ln|x|| + y$$

$$P(x,y) = \frac{\partial F}{\partial x} \qquad Q(x,y) = \frac{\partial}{\partial y} F$$

Implicit solutions ~~Form~~ $F(x,y) = C$

$$F(x,y) = \int \frac{y}{x\ln|x|}\,dx + g(y) = y\int \frac{1}{x\ln|x|}\,dx + g(y)$$

$$\int \frac{1}{x\ln|x|}\,dx = \ln|\ln|x||$$

$$F(x,y) = y\ln|\ln|x|| + g(y)$$

$$Q(x,y) = \ln|\ln|x|| + y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[y\ln|\ln|x|| + g(y)\right] =$$

$$= \ln|\ln|x|| + g'(y)$$

Using $\qquad Q(x,y) = \frac{\partial F}{\partial y}$

$$\underbrace{\ln|\ln|x|| + y}_{Q(x,y)} = \frac{\partial}{\partial y} F = \frac{\partial}{\partial y}\left[y \ln|\ln|x|| + g(y) \right] = \ln|\ln|x|| + g'(y)$$

$$\cancel{\ln|\ln|x||} + y = \cancel{\ln|\ln|x||} + g'(y)$$

$$g'(y) = y \qquad g(y) = \int y\, dy + c'$$

$$= \frac{1}{2} y^2 + c'$$

$$F(x,y) = y \ln|\ln|x|| + \frac{1}{2} y^2 + c'$$

Implicit solution $\quad F(x,y) = c$

$$\boxed{y \ln|\ln|x|| + \frac{1}{2} y^2 = c''}$$

c'' is an arbitrary constant

$c'' = c - c'$

Exam 2022-2023

Solve the ODE

$$\ddot{y} + 2\dot{y} + 3y = 25 = f(t)$$

Corresponding homogeneous ODE $\ddot{y} + 2\dot{y} + 3y = 0$

Characteristic equation: $\lambda^2 + 2\lambda + 3 = 0$

Roots: $\lambda = \dfrac{-2 \pm \sqrt{4-12}}{2} = \dfrac{-2 \pm \sqrt{-8}}{2} = \dfrac{-2 \pm i2\sqrt{2}}{2} = \begin{cases} -1+i\sqrt{2} \\ -1-i\sqrt{2} \end{cases}$

$\lambda_1 = -1 + i\sqrt{2}$ $\qquad \alpha = \operatorname{Re}\lambda_1 = -1$

$\lambda_2 = -1 - i\sqrt{2}$ $\qquad \beta = \operatorname{Im}\lambda_1 = \sqrt{2}$

General solution of the homogeneous ODE

$$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

c_1, c_2 are complex conjugate arbitrary constants.

$$y_h(t) = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$$

$A, B \in \mathbb{R}$ arbitrary constant

$$y_h(t) = e^{-t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right)$$

The solution to the inhomogeneous ODE

$$y(t) = y_h(t) + y_p(t)$$

General solution to the homogeneous ODE

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} =$$

$$y_h(t) = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right) \quad \text{where } \lambda_1 = \alpha + i\beta$$

$$y_h(t) = e^{-t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right) \quad \text{where } A, B \text{ are arbitrary constants.}$$

The solution to the inhomogeneous ODE reads

$$y(t) = y_h(t) + y_p(t)$$

Using the educated guess method. $f(t) = 25 = p(t)\, e^{\tilde{\alpha} t}$

$\tilde{\alpha} = 0$

$25 = p(t)$ polynomial of degree 0

$$y_p(t) = q(t) = d_0 \qquad \dot{y}_p(t) = 0 \qquad \ddot{y}_p = 0$$

Inserting $y_p(t) = d_0$ $\dot{y}_p(t) = \ddot{y}_p(t) = 0$ into the ODE

$$\underbrace{\ddot{y}_p}_{=0} + 2\underbrace{\dot{y}_p}_{=0} + 3y_p = 25 \qquad 3d_0 = 25$$

$$d_0 = \frac{25}{3} = y_p(t)$$

The general solution of the inhomogeneous ODE is

$$y(t) = y_h(t) + y_p(t) = e^{-t}\left(A\cos\sqrt{2}t + B\sin\sqrt{2}t\right) + \frac{25}{3}$$

where A, B are arbitrary constants.