Lyapunov function

We consider a autonomous system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \qquad (1)$$

having an equilibrium point at $Y_*(t) = \begin{pmatrix} a \\ b \end{pmatrix}$

A Lyapunov function $V(y_1, y_2)$ is a continuously differentiable function defined for $|Y(t) - Y_*(t)| < R$ such that

1. $V(a,b) = 0$
2. $V(y_1, y_2) > 0$ for $(y_1, y_2) \neq (a,b)$
3. $D_f(V) \leq 0$ for $(y_1, y_2) \neq (a,b)$

If a Lyapunov function exist then $Y_*(t)$ is Lyapunov stable solution

If the Lyapunov function satisfies

3.' $D_f(V) < 0$ for $(y_1, y_2) \neq (a,b)$

Then $Y_*(t)$ is asymptotically stable

Gradient flow and potential function

Consider a continuously differentiable function $V(y_1, y_2)$ satisfying

1. $V(a,b) = 0$
2. $V(y_1, y_2) > 0 \qquad \forall\, (y_1, y_2) \neq (a,b)$

If the dynamical system (1) can be written as

$$\dot{y}_1 = -\frac{\partial V}{\partial y_1}$$

$$\dot{y}_2 = -\frac{\partial V}{\partial y_2}$$

then the dynamical system is called a gradient flow

$V(y_1, y_2)$ is a Lyapunov function called potential

(a,b) is an equilibrium point of the gradient flow

(a,b) is Lyapunov stable

Proof Since $V(a,b)=0$ and $V(y_1,y_2)>0$ for $(y_1,y_2)\neq(a,b)$

$y_*(t)=\begin{pmatrix} a \\ b \end{pmatrix}$ is a minimum of V

hence $\left(\left.\frac{\partial V}{\partial y_1}\right|_{(a,b)}, \left.\frac{\partial V}{\partial y_2}\right|_{(a,b)} \right) = (0,0)$

It follows $\dot{y}_1 = -\left.\frac{\partial V}{\partial y_1}\right|_{(a,b)} = 0$ and $\dot{y}_2 = -\left.\frac{\partial V}{\partial y_2}\right|_{(a,b)} = 0$

hence $y_*(t)=\begin{pmatrix} a \\ b \end{pmatrix}$ is an equilibrium point.

Let us verify that V is a viable Lyapunov function:

1. $V(a,b)=0$
2. $V(y_1,y_2)>0$ for $(y_1,y_2)\neq(a,b)$
3. $D_f(V) = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 = -\left(\frac{\partial V}{\partial y_1}\right)^2 - \left(\frac{\partial V}{\partial y_2}\right)^2$

Using $\dot{y}_1 = -\frac{\partial V}{\partial y_1}$

$\dot{y}_2 = -\frac{\partial V}{\partial y_2}$

$$\Rightarrow D_f(V) = -\left[\left(\frac{\partial V}{\partial y_1}\right)^2 + \left(\frac{\partial V}{\partial y_2}\right)^2\right] \leq 0$$

V is a Lyapunov function

The equilibrium solution is Lyapunov stable

Linear stability theorem

Consider the non-linear automomous system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \qquad (1)$$

and its linearised system

$$\dot{Y} = AY$$

around the equilibrium solution $Y_*(t) = \begin{pmatrix} a \\ b \end{pmatrix}$.

Take for simplicity $Y_*(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and indicate with λ_1, λ_2 the two eigenvalues of A.

1. If $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) < 0$

   the zero solution is ASYMPTOTICALLY STABLE for the system (1)

2. If $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) > 0$

   the zero solution is UNSTABLE for the system (1)

3. If $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) = 0$

   the zero solution can be either STABLE or UNSTABLE

If the linearised system is ASYMPTOTICALLY STABLE

the non-linear system is also ASYMPTOTICALLY STABLE

If the linearised system is UNSTABLE

the non-linear system is also UNSTABLE

If the linearised system is STABLE but not asymptotically stable

the so called linear stability analysis fails and in order to establish the stability of the solution it is necessary to consider the non-linear terms.

Summary of second half of the module

Week 8 Dynamical systems, equilibrium points

Linearisation of non linear systems

Week 9-10 We have solved linear systems of ODEs with constant coefficients. Phase portraits

Week 11 Stability of solutions. Gradient flows. Lyapunov functions. Relation between the stability of equilibrium point for non-linear system of ODEs and for their corresponding linearised system of ODEs.