Stability criteria for linear system of ODEs

We consider the linear system

$$\dot{Y} = AY \quad (2)$$

where A is a 2×2 matrix with constant coefficients and eigenvalues

λ_1, λ_2,

This system admits the zero solution $Y(t) = (0,0)^T$ or

otherwise $(y_1, y_2) = (0,0)$ is an equilibrium point.

Theorem Define $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2)$ then the zero solution

$Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

1. UNSTABLE if $s > 0$
2. STABLE (Lyapunov stable) if $s = 0$
3. ASYMPTOTICALLY STABLE if $s < 0$

The theorem is valid also for $\lambda_1 = \lambda_2$. Let us derive the result when $\lambda_1 \neq \lambda_2$.

Stability criteria for linear systems of ODEs

We will consider the linear autonomous system

$$\dot{Y} = AY \qquad (2)$$

where A is a 2×2 matrix, with constant coefficients and eigenvalues λ_1, λ_2

The system admits the zero solution $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as an ~~the~~ equilibrium point, i.e. $(y_1, y_2) = (0,0)$

Theorem Define $s = \max\left(\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)\right)$ then the zero solution $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

1. UNSTABLE if $s > 0$

2. ASYMPTOTICALLY STABLE if $s < 0$

3. LYAPUNOV STABLE if $s = 0$

   ~~(but not asymp~~

The theorem is valid also for $\lambda_1 = \lambda_2$.

Let us derive the result when $\lambda_1 \neq \lambda_2$

c) If $s = \max(\lambda_1, \lambda_2) = 0$ If $\lambda_1 = 0$ $\lambda_2 < 0$

$$e^{\lambda_1 t} = 1$$

$$e^{\lambda_2 t} \to 0 \quad \text{as } t \to \infty$$

The zero solution is STABLE

② λ_1, λ_2 complex conjugate

$\lambda_1 = \alpha + i\beta$

$\alpha, \beta \in \mathbb{R}$ $\beta > 0$

$\lambda_2 = \alpha - i\beta$

$s = \max(\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2) = \alpha$

Recall $e^{\lambda_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$

$$e^{\lambda_1 t} = e^{\alpha t}(\cos\beta t + i \sin\beta t)$$

The general solution to (2)

$$Y(t) = e^{\alpha t} \underbrace{\left(D_1 e^{i\beta t} u_1 + D_2 e^{-i\beta t} u_2\right)}_{\text{oscillatory part}}$$

<u>Derivation</u> We observe that $Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium solution to Eqs. (2)

Indeed $\dot{Y} = AY$ for $Y(0) = 0$

$\dot{Y} = A\,Y(0) = 0$

The general solution to Eq. (2) when $\lambda_1 \neq \lambda_2$ is

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

① $\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad D_1, D_2 \in \mathbb{R}$

a) If $s = \max(\lambda_1, \lambda_2) > 0$ UNSTABLE NODE, SADDLE

$e^{\lambda_1 t} \to \infty$

or as $t \to \infty$

$e^{\lambda_2 t} \to \infty$

The trajectories from I.C. $Y(0)$ arbitrarily close to the origin go to infinity.

The zero solution is UNSTABLE

b) If $s = \max(\lambda_1, \lambda_2) < 0$ STABLE NODE

$e^{\lambda_1 t} \to 0 \quad e^{\lambda_2 t} \to 0 \quad$ as $t \to \infty$

The zero solution is ASYMPTOTICALLY STABLE

② λ_1, λ_2 complex conjugate

$$\lambda_1 = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R} \quad \beta > 0$$
$$\lambda_2 = \alpha - i\beta$$

$$e^{\lambda_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$$

$$s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) = \alpha$$

The general solution reads

$$y(t) = e^{\alpha t} \underbrace{\left(D_1 e^{i\beta t} u_1 + D_2 e^{-i\beta t} u_2 \right)}_{\text{oscillatory part}}$$

- If $s = \alpha > 0$ $\qquad e^{\alpha t} \to \infty$ as $t \to \infty$ UNSTABLE FOCUS

  The zero solution is UNSTABLE

- If $s = \alpha < 0$ $\qquad e^{\alpha t} \to 0$ as $t \to \infty$ STABLE FOCUS

  The zero solution is ~~asy~~ ASYMPTOTICALLY STABLE

- If $s = \alpha = 0$ $\qquad e^{\alpha t} = 1$ CENTRE

  The zero solution is STABLE

What can we say about the stability of the equilibrium point for a NON-LINEAR system of ODE?

a) If $s = \alpha > 0$ $\quad e^{\alpha t} \to \infty$ as $t \to \infty$ UNSTABLE FOCUS (SPIRAL OUT)

The zero solution will be UNSTABLE

b) If $s = \alpha < 0$ $\quad e^{\alpha t} \to 0$ as $t \to \infty$ STABLE FOCUS (SPIRAL IN)

The zero solution will be ASYMPTOTICALLY STABLE

c) If $s = \alpha = 0$ $\quad e^{\alpha t} = 1$ CENTRE

The zero solution will be STABLE.

What can we say about the stability of the equilibrium point of NON-LINEAR systems of ODEs?

Directional derivative - Orbital derivative.

Consider an autonomous system of ODEs:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \quad (1)$$

Suppose that $Y(t)$ is a solution to (1) with $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

For any continuosly differentiable function $V(\tilde{y}_1, \tilde{y}_2)$ we can define its value at a given time along the solution $Y(t)$ given by

$$v = V\big(y_1(t), y_2(t)\big)$$

The directional derivative or orbital derivative of V along $\vec{f} = \big(f_1(y_1, y_2), f_2(y_1, y_2)\big)$ is given by

$$\boxed{D_f(V) = \dot{v} = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2}$$

Using $\dot{y}_1 = f_1(y_1, y_2)$ & $\dot{y}_2 = f_2(y_1, y_2)$ we have

$$\boxed{D_f(V) = \frac{\partial V}{\partial y_1} f_1(y_1, y_2) + \frac{\partial V}{\partial y_2} f_2(y_1, y_2)}$$

Lyapunov function

Theorem : Lyapunov stability theorem

Let $Y_*(t) = (a, b)^T$ be an equilibrium solution to (1)

Assume that inside a circle $0 < |Y(t) - Y_*(t)| < R$

there exist a continuously differentiable function

$V(y_1, y_2)$ satisfying

1. $V(a, b) = 0$
2. $V(y_1, y_2) > 0$ for any $(y_1, y_2) \neq (a, b)$
3. $D_f(V) \leq 0$ for any $(y_1, y_2) \neq (a, b)$

$V(y_1, y_2)$ is called Lypunov function of (1)

The solution $Y_*(t) = (a, b)^T$ is Lyapunov STABLE

Theorem Lypunov asymptotic stability theorem

If the Lyapunov function satisfies 1., 2., 3.'

where 3'.

3.' $D_f(V) < 0$ for any $(y_1, y_2) \neq (a, b)$

$\Rightarrow$ The solution $Y_*(t) = (a, b)^T$ is ASYMPTOTICALLY STABLE