Tutorial week 11

Consider the non-linear system

$$\begin{cases} \dot{y}_1 = y_2 + a y_1 - y_1^5 = f_1(y_1, y_2) \\ \dot{y}_2 = -y_1 - y_2^5 = f_2(y_1, y_2) \end{cases} \qquad a \in \mathbb{R} \qquad (1)$$

(A) Linearise the system around the equilibrium point $(0,0) = (y_1, y_2)$

(B) For which values of a the linearised system displays a stable focus?

(C) Put $a = 0$ and check that $V(y_1, y_2) = y_1^2 + y_2^2$ is a Lyapunov function of (1). Draw conclusion on the stability of the non-linear system for $a = 0$

(A) - Check that $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point of (1)

$f_1(0,0) = 0$ ✓

$f_2(0,0) = 0$ ✓

$\rightarrow$ $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point

The linearised system is $\dot{Y} = AY$ with

$$A = \begin{pmatrix} \left.\dfrac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_1}{\partial y_2}\right|_{(0,0)} \\ \left.\dfrac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_2}{\partial y_2}\right|_{(0,0)} \end{pmatrix}$$

$$f_1(y_1,y_2) = y_2 + a y_1 - y_1^5 \qquad f_2(y_1,y_2) = -y_1 - y_2^5$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = \left. a - 5y_1^4 \right|_{(0,0)} = a \qquad \left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = -1 \qquad \left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \left. -5y_2^4 \right|_{(0,0)} = 0$$

$$A = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

(B) Now we need to impose that the fixed point is a stable focus.

We need to impose that A has complex conjugate eigenvalues

$$\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta \qquad \alpha, \beta \in \mathbb{R} \quad \beta \neq 0$$

$$\alpha < 0$$

Let us calculate the eigenvalues of A

$$\det(A - \lambda Id) = \det\begin{pmatrix} a-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (a-\lambda)(-\lambda) + 1 = 0$$

$\lambda^2 - a\lambda + 1 = 0 \Rightarrow$ Roots

$$\lambda = \frac{a \pm \sqrt{a^2-4}}{2} \quad \begin{cases} \lambda_1 = \dfrac{a + \sqrt{a^2-4}}{2} \\[2ex] \lambda_2 = \dfrac{a - \sqrt{a^2-4}}{2} \end{cases}$$

λ_1, λ_2 are complex conjugate if and only if

$$\Delta = a^2 - 4 < 0$$

$$a^2 < 4 \quad \Rightarrow \quad -2 < a < 2$$

In order to have a stable focus we need to impose additionally

$$\alpha < 0$$

If $-2 < a < 2$

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$

$$\lambda_1 = \frac{a}{2} + i\frac{\sqrt{4-a^2}}{2} \qquad \lambda_2 = \frac{a}{2} - i\frac{\sqrt{4-a^2}}{2}$$

$$\alpha = \frac{a}{2} \qquad \beta = \frac{\sqrt{4-a^2}}{2}$$

Let us impose $\alpha < 0 \Rightarrow \frac{a}{2} < 0 \Rightarrow a < 0$

In order to satisfy $-2 < a < 2$ and $a < 0$

we require $\boxed{-2 < a < 0}$

The linearised system displays a stable focus if and only if

$$-2 < a < 0$$

(c) Put $a=0$. Check that $V(y_1,y_2) = y_1^2 + y_2^2$ is a Lypunov function of the non-linear system.

We need to check 3 conditions

1. $V(0,0) = y_1^2 + y_2^2 \Big|_{(0,0)} = 0$ ✓

2. $V(y_1,y_2) = y_1^2 + y_2^2 > 0$ ✓ if $(y_1,y_2) \neq (0,0)$

3'. $D_f(V) = \frac{\partial V}{\partial y_1} \dot{y}_1 + \frac{\partial V}{\partial y_2} \dot{y}_2 = -2\left[y_1^6 + y_2^6\right] < 0$ if $(y_1,y_2) \neq (0,0)$

$$\frac{\partial V}{\partial y_1} = \frac{\partial}{\partial y_1}(y_1^2 + y_2^2) = 2y_1 \qquad \dot{y}_1 = f_1(y_1,y_2) = y_2 - y_1^5$$

$$\frac{\partial V}{\partial y_2} = \frac{\partial}{\partial y_2}(y_1^2 + y_2^2) = 2y_2 \qquad \dot{y}_2 = f_2(y_1,y_2) = -y_1 - y_2^5$$

$$D_f(V) = 2y_1\left[y_2 - y_1^5\right] + 2y_2\left[-y_1 - y_2^5\right] =$$

$$= \cancel{2y_1y_2} - 2y_1^6 - \cancel{2y_1y_2} - 2y_2^6 = -2\left[y_1^6 + y_2^6\right]$$

$V(y_1,y_2)$ is a Lyapunov function of (1)

The zero solution is ASYMPTOTICALLY STABLE