Queen Mary University of London

MTH5123 Differential Equations

Formative Assessment: Week 10 – Solutions G. Bianconi

I. Practice Problems-Solution of selected questions

A. Determine the type of equilibrium at $y_1 = 0, y_2 = 0$ for the following ODE systems.

A remind of Coursework 8 and the solutions

1) $\dot{y_1} = -\frac{1}{2}y_1 + \frac{5}{2}y_2$, $\dot{y_2} = \frac{5}{2}y_1 - \frac{1}{2}y_2$, $y_1(0) = a, y_2(0) = b$. The solution to this I.V.P. is given by $y_1 = \frac{1}{2}(a+b)e^{2t} + \frac{1}{2}(a-b)e^{-3t}$, $y_2 = \frac{1}{2}(a+b)e^{2t} - \frac{1}{2}(a-b)e^{-3t}$.

2) $\dot{y_1} = -y_1 + 5y_2$, $\dot{y_2} = -y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 4$. The solution to this I.V.P. is given by $y_1 = 10 \sin 2t$, $y_2 = 2 \sin 2t + 4 \cos 2t$.

Solutions:

(1) The fixed point is a saddle, becaues we have the two real eigenvalues, one positive and one negative for this linear ODE systems.

(Revision of Sketching phase portraits: Choosing the initial conditions such that $a = b$ we have $y_1 = y_2 = ae^{2t}$ for any t, and this line defines one of the invariant manifolds. Along this line the motion is away from the origin towards $\pm\infty$ for $t \to \infty$ (for $a > 0$ and $a < 0$, respectively). The second invariant manifold $y_2 = -y_1$ corresponds to the initial conditions $b = -a$. Along this line the motion is towards the origin, i.e., for $t \to \infty$ we have $y_2 = -y_1 \to 0$. For initial conditions away from the two invariant manifolds we have asymptotically $y_2 \approx y_1 \approx \frac{1}{2}(a+b)e^{2t}$ for $t \to \infty$. This means the typical trajectories are hyperbolic-like curves whose tangent is parallel asymptotically to the $y_2 = y_1$ direction for $t \to \infty$ and is parallel to the $y_2 = -y_1$ direction for $t \to -\infty$; see the figure to the end (left).)

(2)The fixed point is a centre, because the eigenvalues are two complex numbers with the real part (or TrA) equals to 0.

In addition, according to the solution to this IVP, the trajectories must be ellipses; see the figure below for the particular trajectory passing through the given initial conditions.

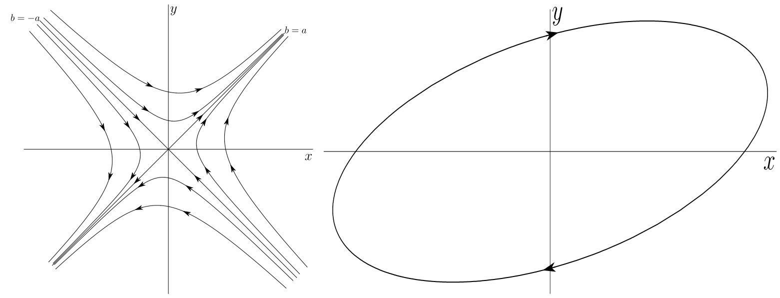

Figure 1: Left: The phase portrait for problem A1) featuring a saddle. Right: The trajectory solving the initial value problem A2). The equilibrium at (0,0) is a center.

B. Determine the general solution, sketch the phase portraits and determine the type of equilibrium at $y_1 = 0, y_2 = 0$ for the following systems of linear differential equations:

1) $\dot{y_1} = -y_1 + 6y_2\ ,\ \dot{y_2} = -3y_1 + 8y_2$

Solution: First we rewrite the system in the matrix form $\dot{\mathbf{y}} = A\mathbf{u}$, where $A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$. Next we obtain the characteristic equation and determine the eigenvalues:

$$(-1-\lambda)(8-\lambda) + 18 = \lambda^2 - 7\lambda + 10 = 0\ ,\ \lambda_1 = 5,\ \lambda_2 = 2\ .$$

Thus, the equilibrium at $(0,0)$ is an unstable node source, because both eigenvalues are positive. Then we determine the eigenvector components for $\lambda_1 = 5$:

$$\begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 5\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -p_1 + 6q_1 = 5p_1 \\ -3p_1 + 8q_1 = 5q_1 \end{matrix}\ ,$$

which gives $p_1 = q_1$ so that the corresponding eigenvector can be chosen to $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, we find that the eigenvector corresponding to $\lambda_1 = 2$ is given by $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$. Finally, the general solution to the system of ODEs is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

or in components

$$y_1(t) = C_1 e^{5t} + C_2 e^{2t}\ ,\ y_2(t) = C_1 e^{5t} + \frac{1}{2}C_2 e^{2t}\ .$$

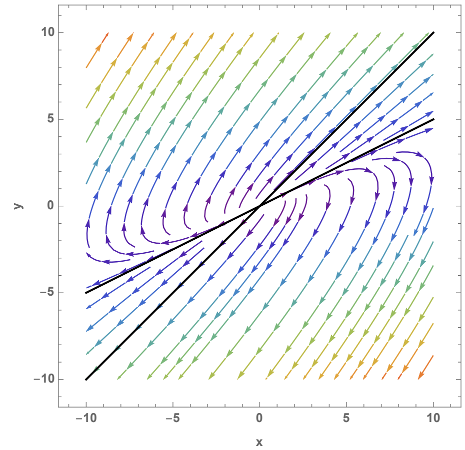

Figure 2: The phase portrait for problem B1) featuring an unstable node source.

2) $\dot{y_1} = -y_1 + y_2$, $\dot{y_2} = y_1 - y_2$

Solution: We rewrite the system in the matrix form $\dot{\mathbf{x}} = A\mathbf{u}$ with $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. The characteristic equation is $(-1-\lambda)(-1-\lambda)-1 = \lambda^2+2\lambda = 0, \quad \lambda_1 = 0,\ \lambda_2 = -2$. **Thus, the equilibrium at $(0,0)$ is stable, as one eigenvalue is negative and the other is zero.** The corresponding eigenvector can be chosen as $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Finally, the general solution to the system of ODEs is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or in components

$$y_1(t) = C_1 + C_2 e^{-2t} \ , \ y_2(t) = C_1 - C_2 e^{2t} \ .$$

Note: In the above system, we have two real eigenvalues and one eigenvalue is zero. All points on the line of the eigenvector $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ obtained under $\lambda = 0$ are all equilibrium points. You can check this by the ODE, when $y_2 = y_1$ both $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. The equilibrium at $(0,0)$ is stable (see the phase portraits below) but not a stable node sink.

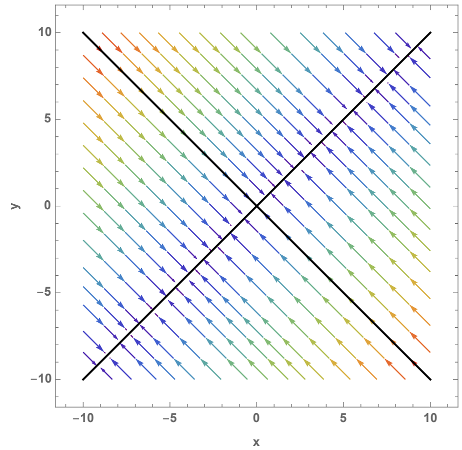

Figure 3: The phase portrait for problem B2) featuring a stable equilibrium.

3) $\dot{y_1} = -4y_1 - 8y_2$, $\dot{y_2} = 4y_1 + 4y_2$

Solution: We rewrite the system in the matrix form $\dot{\mathbf{y}} = A\mathbf{u}$ with $A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix}$. The characteristic equation is $(-4-\lambda)(4-\lambda)+32 = \lambda^2+16 = 0$ which yields two complex-conjugate eigenvalues $\lambda_1 = 4i$, $\lambda_2 = -4i$. **we can know that the equilibrium at $(0,0)$ is a center, because the two eigenvalues are complex and their real parts both equal to zero.** The eigenvector components for $\lambda_1 = 4i$ are obtained by

$$\begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 4i \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -4p_1 - 8q_1 = 4ip_1 \\ 4p_1 + 4q_1 = 4iq_1 \end{matrix} .$$

This gives $q_1 = -\frac{1+i}{2}p_1$ so that the corresponding eigenvector can be chosen, for example, as $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$. We can immediately conclude that the eigenvector corresponding to $\lambda_2 = -4i$ can be chosen in the complex conjugate form $\mathbf{u}_2 = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$. Finally, the general solution to the system of ODEs can be written in the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{4it} \begin{pmatrix} 2 \\ -1-i \end{pmatrix} + C_2 e^{-4it} \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$$

or in components

$$y_1(t) = 2C_1e^{4it} + 2C_2e^{-4it} , \; y_2(t) = (-1-i)C_1e^{4it} + (-1+i)C_2e^{-4it} .$$

For the initial conditions $y_1(0) = a$ and $y_2(0) = b$, where (a,b) can be any point in the phase plane

$$y_1(0) = C_1 + C_2 = a/2 , \; y_2(0) = -1(C_1+C_2) + (C_2 - C_1)i = b$$

$$\Rightarrow \; C_1 = \frac{a}{4} + \frac{2b+a}{4}i, \; C_2 = \frac{a}{4} - \frac{2b+a}{4}i,$$

Thus the solution to this IVP will be

$$y_1(t) = 2(\frac{a}{4} + \frac{2b+a}{4}i)e^{4it} + 2(\frac{a}{4} - \frac{2b+a}{4}i)e^{-4it} = a\cos 4t \ - (2b+a)\sin 4t,$$

$$y_2(t) = (-1-i)(\frac{a}{4} + \frac{2b+a}{4}i)e^{4it} + (-1+i)(\frac{a}{4} - \frac{2b+a}{4}i)e^{-4it} = (b+a)\sin 4t + b\cos 4t.$$

As the phase portrait is a centre. If we plot one ellipse (one trajectory), then all other trajectories are just a set of nested ellipses around the equilibrium point in this linear system $(0,0)$. Thus, pick any values for (a,b), e.g. $(1,0)$, the solution to this IVP becomes

$$y_1(t) = \cos 4t \ - \sin 4t,$$

$$y_2(t) = \sin 4t.$$

You can plot the ellipse by chose t=0, $\frac{\pi}{16}, \frac{2\pi}{16}, \frac{4\pi}{16}, \frac{5\pi}{16}, \frac{6\pi}{16}...$, which is the black ellipse in the Figure 4. Then you draw rest trajectories accordingly.

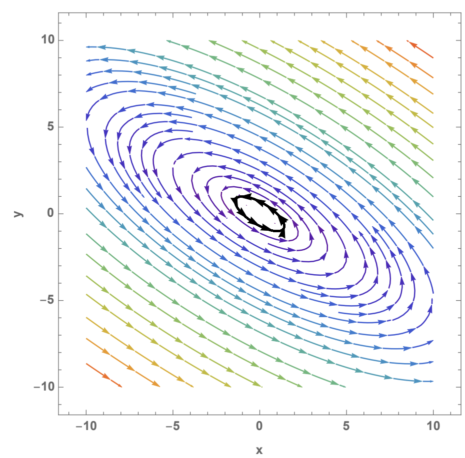

Figure 4: The phase portrait for problem B3) featuring a center.

If you transform the $y_1 y_2$ coordinates to new coordinates as described in our lecture, the trajectories will becomes counterclockwise circles around $(0,0)$.

C. Determine the type of fixed point for the dynamical systems

$$\dot{y_1} = 4y_2 \ , \ \dot{y_2} = -y_1.$$

Then determine the solutions of the corresponding initial value problems for the general initial conditions $y_1(0) = a, y_2(0) = b$. Finally sketch the phase portraits in the (y_1, y_2) phase plane.

Solution. The matrix associated with this system is given by $A = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$. The characteristic equation is $\lambda^2 + 4 = 0$ with two complex conjugate roots $\lambda_1 = 2i$, $\lambda_2 = -2i$.

As the eigenvalues are complex conjugate and their real parts equal to zero, the corresponding fixed point is a center.

The eigenvector corresponding to $\lambda_1 = 2i$ can be found from

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2i \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad \Rightarrow q_1 = \frac{i}{2} p_1$$

so that the eigenvectors are $\mathbf{u}_1 = \begin{pmatrix} 1 \\ \frac{i}{2} \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -\frac{i}{2} \end{pmatrix}$. The general solution has the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{2it} \begin{pmatrix} 1 \\ \frac{i}{2} \end{pmatrix} + C_2 e^{-2it} \begin{pmatrix} 1 \\ -\frac{i}{2} \end{pmatrix} .$$

The initial conditions yield

$$y_1(0) = C_1 + C_2 = a \ , \ y_2(0) = \frac{i}{2}(C_1 - C_2) = b \ \Rightarrow \ C_1 = \frac{1}{2}(a - 2ib), \ C_2 = \frac{1}{2}(a + 2ib) y_2 =$$

so that the solution to the general initial value problem is given by

$$y_1 = \frac{1}{2}(a - 2ib)e^{2it} + \frac{1}{2}(a + 2ib)e^{-2it} = a \cos 2t + 2b \sin 2t,$$

and similarly

$$y_2 = \frac{i}{4}(a - 2ib)e^{2it} - \frac{i}{4}(a + 2ib)e^{-2it} = -\frac{a}{2} \sin 2t + b \cos 2t \ .$$

We notice that $y_1^2 + 4y_2^2 = a^2 + 4b^2$ describing ellipses in phase space. The check the direction of the trajectories (arrows), we can pick up any initial point, for example, $y_1(0) = 1, y_2(0) = 0$, then the tangent vector at this point is $\begin{pmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \end{pmatrix} = A \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus the trajectory starting at $(1, 0)$ will move towards negative y values (clockwise).

Note the trajectories here are clockwise as it is in the original $y_1 y_2$ coordinates. If you transform the $y_1 y_2$ coordinates to new coordinates as described in our lecture, the trajectories will becomes counterclockwise circles around $(0, 0)$.

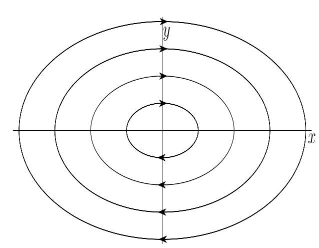

Figure 5: Phase portrait for problem C featuring a centre.

II. Homework

A. Determine the solution of the initial value problem

$$\dot{y_1} = y_1 - 4y_2 \; , \; \dot{y_2} = 4y_1 + y_2 \; , \; y_1(0) = 0 \; , \; y_2(0) = 1 \; , \; t \geq 0$$

and the type of fixed point. Then sketch the trajectory in the (y_1, y_2) phase plane corresponding to the chosen initial values in the specified range of t.

Solution. The matrix associated with the system is given by $A = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}$. The characteristic equation is $\lambda^2 - 2\lambda + 17 = 0$ with two complex conjugate roots $\lambda_1 = 1 + 4i$, $\lambda_2 = 1 - 4i$. The eigenvector corresponding to $\lambda_1 = 1 + 4i$ can be found from

$$\begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = (1 + 4i) \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \;\Rightarrow\; q_1 = -ip_1$$

so that the eigenvectors are $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. The general solution has the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{t+4it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{t-4it} \begin{pmatrix} 1 \\ i \end{pmatrix} .$$

The initial conditions yield

$$y_1(0) = C_1 + C_2 = 0 \; , \; y_2(0) = C_1(-i) + C_2 \, i = 1 \; , \;\Rightarrow\; C_1 = \frac{i}{2}, \; C_2 = \frac{-i}{2}$$

so that the solution to the general initial value problem is given by

$$y_1 = \frac{i}{2} e^{t+4it} - \frac{i}{2} e^{t-4it} = -e^t \sin 4t$$

and similarly

$$y_2 = \frac{i}{2}(-i) e^{t+4it} - \frac{i}{2} i e^{t-4it} = e^t \cos 4t \; .$$

The fixed point is an unstable focus and trajectories are spiraling away from the origin for $t \to \infty$. For the specified initial conditions the initial tangent vector to the trajectory is $\dot{y_1}(0) = -4, \dot{y_2}(0) = 1$ so that the rotation goes anticlockwise; see the sketch below.

III. Applications involving Dynamical Systems

A. Using the relation between charge and current given by $I = dQ/dt$, rewrite the following

$$L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t)$$

as a second order equation in the charge Q. Use this to obtain an ODE for the current I as

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \dot{E}(t).$$

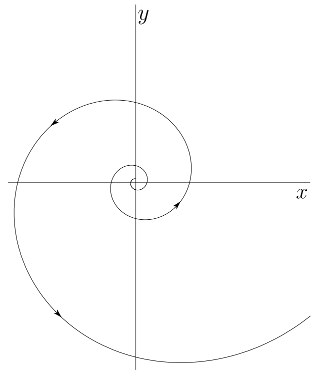

Figure 6: The trajectory solving this initial value problem) with an unstable focus.

Solution. Let differentiation with respect to t be denoted by a dot (Newton's notation for independent variable as time t). Then we have $I = \dot{Q}$ and consequently, $\dot{I} = \ddot{Q}$. The given equation can be rewritten in terms of Q as

$$E(t) = L\frac{dI}{dt} + RI + \frac{1}{C}Q = L\dot{I} + RI + \frac{1}{C}Q = L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q,$$

which is a second-order equation in Q.

We can either differentiating this 2nd-order ODE or the original 1st-order ODE over t, and using the fact that $\dot{Q} = I$, we obtain

$$L\dddot{Q} + R\ddot{Q} + \frac{1}{C}\dot{Q} = L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{E}(t),$$

which is a second order equation in I.

B. Assuming the system is closed and $\dot{E}(t) = 0$, write the second order equation in I as a system of two first order equations using $y_1 = I$ and $y = dI/dt$. Show that $y_1 = 0$, $y_2 = 0$ is a critical point.

Solution. Given $\dot{E}(t) = 0$, we have $L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$. Following the suggested definitions for x and y,

$$\begin{aligned} y_1 = I \quad &\Rightarrow \quad \dot{y}_1 = \dot{I} = y \\ y_2 = \dot{I} \quad &\Rightarrow \quad \dot{y}_2 = \ddot{I} = \frac{1}{L}\left(-R\dot{I} - \frac{1}{C}I\right) = \frac{1}{L}\left(-Ry - \frac{1}{C}x\right). \end{aligned}$$

Rewriting the equation as a linear system, we find

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We immediately see that $(x, y) = (0, 0)$ is a critical point of the system since both the left and right hand sides vanish for this solution.