Tutorial week 10

Consider the dynamical system

$$\begin{cases} \dot{y}_1 = y_1 + e^{y_2} - \cos y_2 & = f_1(y_1, y_2) \\ \dot{y}_2 = 3y_1 - \sin y_2 & = f_2(y_1, y_2) \end{cases} \qquad (1)$$

- Linearise the system of ODEs around the equilibrium point $(y_1, y_2) = (0,0)$
- Establish the nature of the phase portrait of the linearised system
- Sketch the phase portrait of the linearised system

① Check that $(y_1, y_2) = (0,0)$ is an equilibrium point of (1)

$\Leftrightarrow \quad f_1(0,0) = f_2(0,0) = 0$

$$f_1(0,0) = 0 + \underbrace{e^0}_{=1} - \underbrace{\cos 0}_{=1} = 0 \quad ✓$$

(0,0) is an equilibrium point for (1)

$$f_2(0,0) = 0 - \sin 0 = 0 \quad ✓$$

② ~~Linea~~ The linearised system reads $\dot{Y} = AY$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} \left.\dfrac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_1}{\partial y_2}\right|_{(0,0)} \\ \left.\dfrac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_2}{\partial y_2}\right|_{(0,0)} \end{pmatrix}$$

$f_1(y_1,y_2) = y_1 + e^{y_2} - \cos y_2$

$f_2(y_1,y_2) = 3y_1 - \sin y_2$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left.\left(y_1 + e^{y_2} - \cos y_2\right)\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left.\left(y_1 + e^{y_2} - \cos y_2\right)\right|_{(0,0)} = \left.e^{y_2} + \sin y_2\right|_{(0,0)} = e^0 + \sin 0 = 1$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left.\left(3y_1 - \sin y_2\right)\right|_{(0,0)} = 3$$

$$\left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left.\left(3y_1 - \sin y_2\right)\right|_{(0,0)} = \left.-\cos y_2\right|_{(0,0)} = -1$$

The linearised system is $\dot{Y} = AY$ where $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

(3) Determine the type of fixed point for the linearised system

Let us determine the eigenvalues of A

$$\det(A - \lambda \mathrm{Id}) = \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) - 3 = 0$$

$\lambda^2 - 1 - 3 = 0 \quad \Rightarrow \lambda^2 = 4 \quad \Rightarrow \lambda_1 = 2 \; , \; \lambda_2 = -2$

$\boxed{\lambda_1, \lambda_2 \in \mathbb{R} \qquad \lambda_1 \neq \lambda_2 \qquad \lambda_1 > 0 \qquad \lambda_2 < 0}$ SADDLE

④ Sketch the phase portrait

<u>Eigenvectors of A</u> $\qquad \lambda_1 = 2 \qquad u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ satisfying $A u_1 = \lambda_1 u_1$

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \qquad \begin{cases} p_1 + q_1 = 2p_1 \\ 3p_1 - q_1 = 2q_1 \end{cases} \Rightarrow \begin{cases} -p_1 + q_1 = 0 \\ 3p_1 - 3q_1 = 0 \end{cases}$$

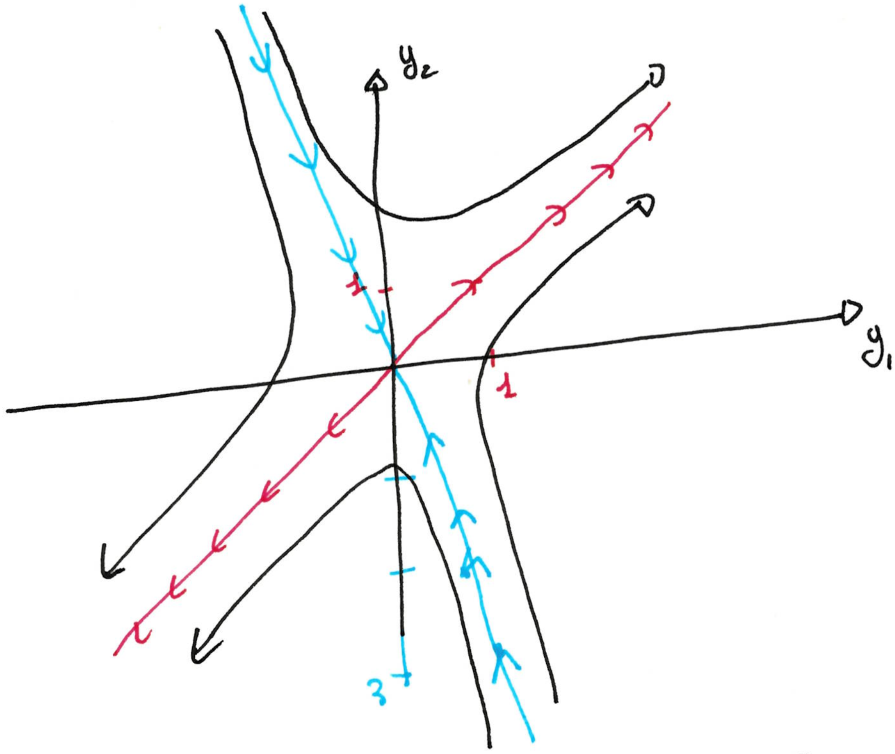
$p_1 = q_1 \qquad$ If $p_1 = 1 \Rightarrow q_1 = 1 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2 \qquad u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$ satisfying $A u_2 = \lambda_2 u_2$

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = -2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \qquad \begin{cases} p_2 + q_2 = -2p_2 \\ 3p_2 - q_2 = -2q_2 \end{cases} \qquad \begin{cases} 3p_2 + q_2 = 0 \\ 3p_2 + q_2 = 0 \end{cases}$$

If $p_2 = 1 \Rightarrow q_2 = -3p_2 = -3 \qquad u_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Sketch the phase portrait



$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_1 = 2 > 0$

UNSTABLE ~~MAN~~ INVARIANT MANIFOLD

$u_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\lambda_2 = -2 < 0$

STABLE INVARIANT MANIFOLD

SADDLE

$\lambda_1, \lambda_2 \in \mathbb{R} \rightsquigarrow \lambda_1 \neq \lambda_2 \rightsquigarrow \lambda_1 > 0 \rightsquigarrow \lambda_2 < 0 \rightsquigarrow \Rightarrow$ $\underline{\text{SADDLE}}$

Consider the dynamical system

$$\dot{y}_1 = -2y_1 - 3y_2 + y_1^5 = f_1(y_1, y_2)$$

$$\dot{y}_2 = y_1 + y_2 - y_2^2 = f_2(y_1, y_2)$$

linearise the dynamical system around the equilibrium point $(0,0)$ and establish the nature of the phase portrait of the linearised system

① Check $(0,0)$ is an equilibrium point $\Longleftrightarrow f_1(0,0) = f_2(0,0) = 0$

$f_1(0,0) = 0$ ✓

$(0,0)$ is an equilibrium point.

$f_2(0,0) = 0$ ✓

② The linearised dynamical system is $\dot{Y} = AY$ with $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

and

$$A = \begin{pmatrix} \left.\dfrac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_1}{\partial y_2}\right|_{(0,0)} \\ \left.\dfrac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\dfrac{\partial f_2}{\partial y_2}\right|_{(0,0)} \end{pmatrix}$$

$$f_1(y_1, y_2) = -2y_1 - 3y_2 + y_1^5$$

$$f_2(y_1, y_2) = y_1 + y_2 - y_2^2$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left(-2y_1 - 3y_2 + y_1^5\right)\Big|_{(0,0)} = -2 + 5y_1^4\Big|_{(0,0)} = -2$$

$$\left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left(-2y_1 - 3y_2 + y_1^5\right)\Big|_{(0,0)} = -3$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left(y_1 + y_2 - y_2^2\right)\Big|_{(0,0)} = 1$$

$$\left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left(y_1 + y_2 - y_2^2\right)\Big|_{(0,0)} = 1 - 2y_2\Big|_{(0,0)} = 1$$

The linearised system is $\dot{Y} = AY$ with $A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$

③ Which is the nature of the fixed point?

$$\det(A - \lambda\, Id) = \det\begin{pmatrix} -2-\lambda & -3 \\ 1 & 1-\lambda \end{pmatrix} = (-2-\lambda)(1-\lambda) + 3 = 0$$

$$\lambda^2 + \lambda - 2 + 3 = \lambda^2 - \lambda + 1 = 0 \qquad \lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \begin{cases} \dfrac{-1+\sqrt{3}\,i}{2} = \lambda_1 \\[2ex] \dfrac{-1-\sqrt{3}\,i}{2} = \lambda_2 \end{cases}$$

λ_1, λ_2 complex conjugate

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$

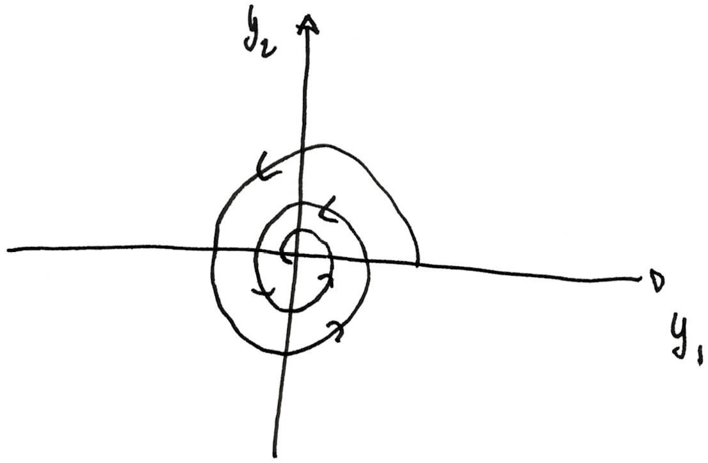
$\alpha = -\frac{1}{2}$

$\beta = \frac{\sqrt{3}}{2}$

$\alpha = \mathrm{Re}\,\lambda_1 = -\frac{1}{2} < 0$

STABLE FOCUS - fixed point

SPIRAL IN - phase portrait.



Sketch of the phase portraist

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