Phase portraits: case complex conjugate eigenvalues.

We consider the linear system of ODEs

$$\dot{Y} = AY$$

where A is a 2×2 real matrix with complex conjugate eigenvalues

$$\lambda_1 = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R} \qquad \beta > 0$$
$$\lambda_2 = \alpha - i\beta$$

and eigenvectors u_1, u_2 chosen to be complex conjugate

The general solution

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2 \qquad (1)$$

In order for Y to be real D_1 and D_2 must be complex conjugate

$$D_1 = \frac{1}{2}(\tilde{a} + i\tilde{b}) \qquad \text{where } \tilde{a}, \tilde{b} \in \mathbb{R}$$
$$D_2 = \frac{1}{2}(\tilde{a} - i\tilde{b}) \qquad \text{arbitrary constants}$$

We consider the basis of vectors v_1, v_2 defined as

$$v_1 = \mathrm{Re}\, u_1 \qquad v_2 = -\mathrm{Im}(u_1)$$

So that
$$u_1 = v_1 - i v_2$$
$$u_2 = v_1 + i v_2$$

v_1 and v_2 are real vectors, and are linearly independent.

The general solution $Y = Y(t)$ in this basis can be written as

$$Y(t) = \tilde{y}_1 v_1 + \tilde{y}_2 v_2$$

with
$$\begin{cases} \tilde{y}_1(t) = e^{\alpha t}\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right) \\ \tilde{y}_2(t) = e^{\alpha t}\left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right) \end{cases} \quad (2)$$

See proof Extra-Material

where $\tilde{a}, \tilde{b} \in \mathbb{R}$ arbitrary constants fixed by the initial condition. Assuming the initial condition is at time $t=0$

$$\begin{cases} \tilde{y}_1(0) = \tilde{a} \\ \tilde{y}_2(0) = \tilde{b} \end{cases}$$

What is the phase portrait of this dynamical system in the coordinates $\tilde{y}_1, \tilde{y}_2$?

Let us consider

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}\left[\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right)^2 + \left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right)^2\right]$$

$$= e^{2\alpha t}\left[\tilde{a}^2\cos^2\beta t + \tilde{b}^2\sin^2\beta t - \cancel{2\tilde{a}\tilde{b}\sin\beta t\cos\beta t}\right.$$
$$\left.\tilde{a}^2\sin^2\beta t + \tilde{b}^2\cos^2\beta t + \cancel{2\tilde{a}\tilde{b}\sin\beta t\cos\beta t}\right]$$

$$= e^{2\alpha t}\left[\tilde{a}^2\underbrace{\left(\cos^2\beta t + \sin^2\beta t\right)}_{1} + \tilde{b}^2\underbrace{\left(\cos^2\beta t + \sin^2\beta t\right)}_{1}\right]$$

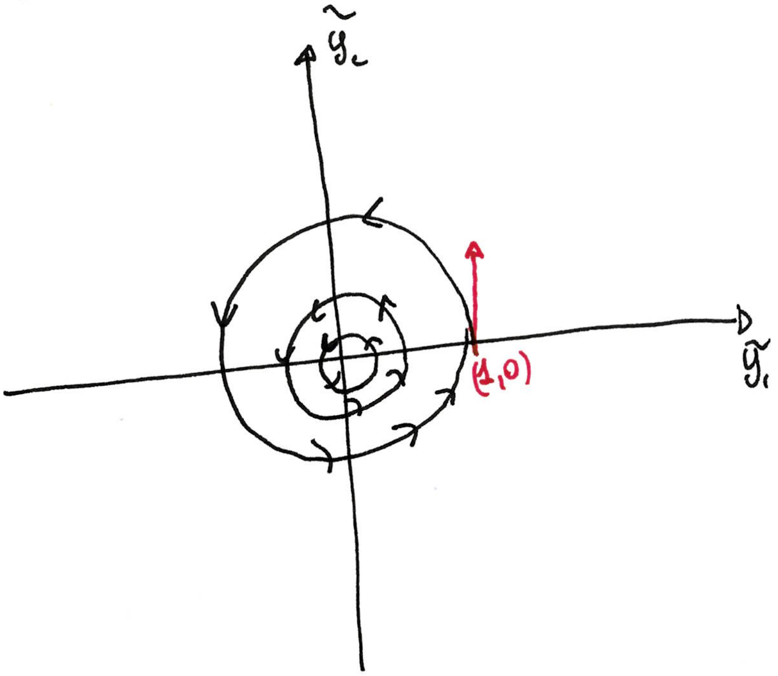
Using $\cos^2\beta t + \sin^2\beta t = 1$

$$\boxed{\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}\left(\tilde{a}^2 + \tilde{b}^2\right)}$$

This equation determines the radial ~~par~~ coordinate of the trajectory.

The trajectories can be classified according to the value of $\alpha = \mathrm{Re}\,\lambda_1$

1 Case I $\alpha = 0$ $(0,0)$ is a CENTRE (neutral stability)



The trajectories satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = (\tilde{a}^2 + \tilde{b}^2) = R^2$$

" $y^2 + x^2 = R^2$ "

The trajectories are circles in the coordinates $(\tilde{y}_1, \tilde{y}_2)$

$\boxed{\text{The direction is counterclockwise}}$

We take $\tilde{y}_1(0) = \tilde{a} = 1$ $(1,0)$

$\tilde{y}_2(0) = \tilde{b} = 0$

Arrows will be determined by the tangent vector $(\dot{\tilde{y}}_1(0), \dot{\tilde{y}}_2(0))$ at time $t=0$. Let us take $\alpha = 0$ $\tilde{a} = 1$ $\tilde{b} = 0$

We know that

$$\begin{cases} \tilde{y}_1(t) = \underset{1}{\tilde{a}} \cos\beta t - \underset{0}{\tilde{b}} \sin\beta t = \cos\beta t \\ \tilde{y}_2(t) = \underset{1}{\tilde{a}} \sin\beta t + \underset{0}{\tilde{b}} \cos\beta t = \sin\beta t \end{cases} \Rightarrow \begin{cases} \dot{\tilde{y}}_1 = -\beta \sin\beta t \\ \dot{\tilde{y}}_2 = \beta \cos\beta t \end{cases}$$

At $t=0$ $\begin{cases} \dot{\tilde{y}}_1(0) = 0 \\ \dot{\tilde{y}}_2(0) = \beta > 0 \end{cases}$

Careful! This discussion is valid only in the new coordinates $\tilde{y}_1, \tilde{y}_2$.

In the original coordinates

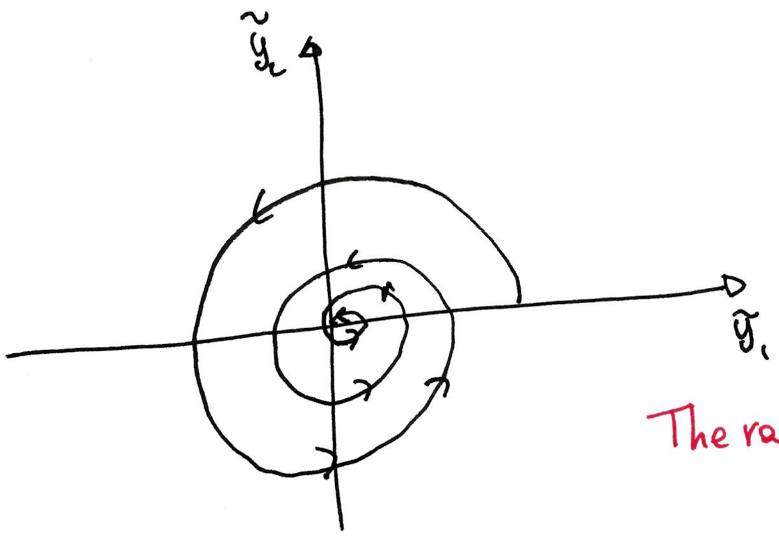
a) the trajectories can be circles or ellipses

b) the direction can be clockwise or couterclockwise

Case II $\alpha < 0$ STABLE FOCUS - STABLE SPIRAL

The trajectories in the coordinates $\tilde{y}_1, \tilde{y}_2$ satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t} (\hat{a}^2 + \hat{b}^2)$$



For $t \to \infty$ $e^{2\alpha t} \to 0$
$(\alpha < 0)$

$$\tilde{y}_1^2 + \tilde{y}_2^2 \to 0$$

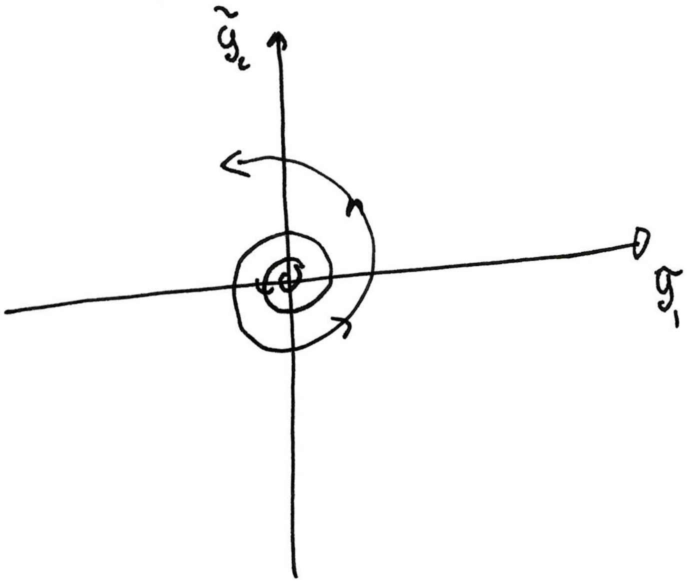
The radial coordinate goes to zero!

Note: The direction in the coordinates $\tilde{y}_1, \tilde{y}_2$ is couterclockwise but in the original coordinates can be either clockwise or couterclockwise.

Case III $\alpha > 0$ UNSTABLE FOCUS, UNSTABLE SPIRAL

The trajectories satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}(\tilde{a}^2 + \tilde{b}^2)$$



At $t \to \infty$ $(\alpha > 0)$

$e^{2\alpha t} \to \infty$

$\tilde{y}_1^2 + \tilde{y}_2^2 \to \infty$

The radial coordinate goes to infinity

In the coordinates $\tilde{y}_1, \tilde{y}_2$ the trajectories go counterclockwise but in the original coordinate they can be ~~eith~~ clockwise or counterclockwise

Extra material : Derivation of Eq (2) from Eq (1)

Recall: Eq (1) is the general solution

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2 \qquad (1)$$

where we assume $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

u_1, u_2 complex conjugate

$u_1 = v_1 - i v_2$ $\qquad u_2 = v_1 + i v_2$

and $D_1 = \frac{1}{2}(\tilde{a} + i\tilde{b})$ $\qquad D_2 = \frac{1}{2}(\tilde{a} - i\tilde{b})$

$\tilde{a}, \tilde{b} \in \mathbb{R}$.

From this Eq. (1) we want to derive Eq (2) given by

$$Y = \tilde{y}_1 v_1 + \tilde{y}_2 v_2 \quad \text{with}$$

$$\begin{cases} \tilde{y}_1 = e^{\alpha t}(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t) \\ \tilde{y}_2 = e^{\alpha t}(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t) \end{cases} \qquad (2)$$

Extra material

Given (1) and the explicit expression of λ_1, λ_2, v_1, v_2, D_1, D_2

We set

$$Y = \frac{1}{2}(\tilde{a} + i\tilde{b})\, e^{(\alpha+i\beta)t} (V_1 - iV_2) +$$

$$+ \frac{1}{2}(\tilde{a} - i\tilde{b})\, e^{(\alpha-i\beta)t} (V_1 + iV_2)$$

Using $c_1 + \bar{c}_1 = 2\,\mathrm{Re}\, c_1$ where $c_1 \in \mathbb{C}$ we set

$$Y = \cancel{2}\,\mathrm{Re}\left[\frac{1}{\cancel{2}}(\tilde{a} + i\tilde{b})\, e^{(\alpha+i\beta)t} (V_1 - iV_2)\right]$$

Using $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i \sin\beta t)$

$$Y = \mathrm{Re}\left[(\tilde{a} + i\tilde{b})\, e^{\alpha t}(\cos\beta t + i\sin\beta t)\,(V_1 - iV_2)\right] =$$

$$= e^{\alpha t}\,\mathrm{Re}\left\{\left[(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t) + i(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t)\right](V_1 - iV_2)\right]$$

Extra material

Using $\mathrm{Re}(c_1 c_2) = (\mathrm{Re}\, c_1)(\mathrm{Re}\, c_2) - (\mathrm{Im}\, c_1)(\mathrm{Im}\, c_2)$ we get

$$Y = e^{\alpha t}\left[(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t)\, v_1 + (\tilde{a}\sin\beta t + \tilde{b}\cos\beta t)\, v_2\right]$$

Since $Y = \tilde{y}_1 v_1 + \tilde{y}_2 v_2$ with $\tilde{y}_1, \tilde{y}_2$ uniquely determined, it follows

$$\begin{cases} \tilde{y}_1 = e^{\alpha t}(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t) \\ \tilde{y}_2 = e^{\alpha t}(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t) \end{cases} \qquad (2)$$

$\square$