Phase portraits: distinct and real eigenvalues

Let us study the phase portraits in the new coordinates $\tilde{y}_1, \tilde{y}_2$!

We know that

$$\begin{cases} \tilde{y}_1 = D_1 e^{\lambda_1 t} \\ \tilde{y}_2 = D_2 e^{\lambda_2 t} \end{cases} \quad (1) \quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2$$

We make two observations:

(A) If $\tilde{y}_1(t=a) = 0$ then $D_1 = 0$

$$0 = \tilde{y}_1(a) = D_1 \underbrace{e^{\lambda_1 a}}_{>0} \Rightarrow D_1 = 0$$

Therefore $\tilde{y}_1(t) = D_1 e^{\lambda_1 t} = 0 \quad \forall t$

$$\boxed{\begin{array}{l} \text{This implies that } \tilde{y}_1 = 0 \text{ is an INVARIANT MANIFOLD} \\ \text{Similarly } \tilde{y}_2 = 0 \text{ is also an INVARIANT MANIFOLD} \end{array}}$$

(B) If $D_1 \neq 0, D_2 \neq 0$ Eq (1) can be also written as

$$\frac{\tilde{y}_1}{D_1} = e^{\lambda_1 t} = \left(e^t\right)^{\lambda_1} \Rightarrow e^t = \left(\frac{\tilde{y}_1}{D_1}\right)^{1/\lambda_1} \quad \text{for } \lambda_1 \neq 0$$

$$\frac{\tilde{y}_2}{D_2} = e^{\lambda_2 t} = \left(e^t\right)^{\lambda_2} = \left[\left(\frac{\tilde{y}_1}{D_1}\right)^{1/\lambda_1}\right]^{\lambda_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\lambda_2/\lambda_1}$$

Therefore the trajectory is given by

$$\boxed{\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\lambda_2/\lambda_1}}$$

for $\lambda_1 \neq 0$ $D_1 \neq 0$, $D_2 \neq 0$

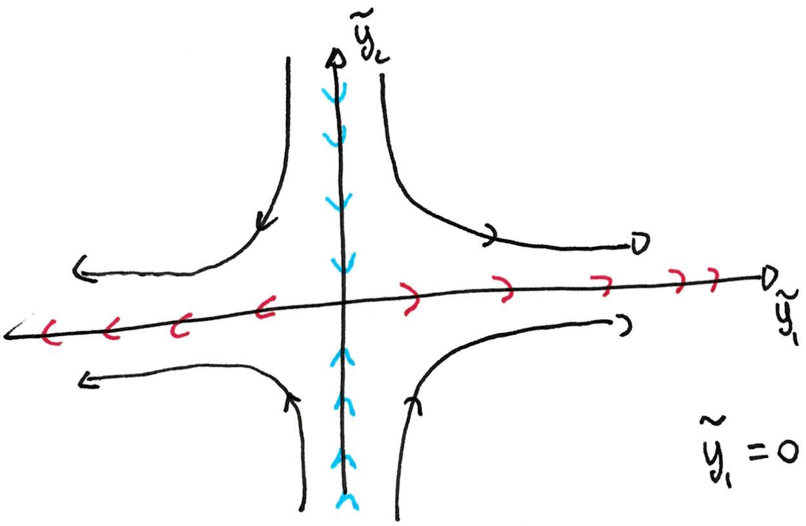
Which are those trajectories ?

$\boxed{1}$ Case I $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$ $\boxed{\lambda_1 > 0, \lambda_2 < 0}$

SADDLE

$$\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\lambda_2/\lambda_1} = \left(\frac{\tilde{y}_1}{D_1}\right)^{-\left|\frac{\lambda_2}{\lambda_1}\right|}$$

For $\left|\frac{\lambda_2}{\lambda_1}\right| = 1$ "$y = x^{-1}$" these are hyperbolas



$\tilde{y}_2 = 0$ $(\lambda_1 > 0)$ UNSTABLE MANIFOLD

$\tilde{y}_1 = 0$ $(\lambda_2 < 0)$ STABLE MANIFOLD

$\tilde{y}_1 = 0 \Rightarrow \tilde{y}_2(t) = D_2 e^{\lambda_2 t}$

As $t \to \infty$ $\tilde{y}_2 \to 0$

$$\begin{cases} \tilde{y}_1 = D_1 e^{\lambda_1 t} & \lambda_1 > 0 \\ \tilde{y}_2 = D_2 e^{\lambda_2 t} & \lambda_2 < 0 \end{cases}$$

$\cancel{(D_1, D_2) \neq (0,0)}$

$D_1 \neq 0$
$D_2 \neq 0$

As $t \to \infty$

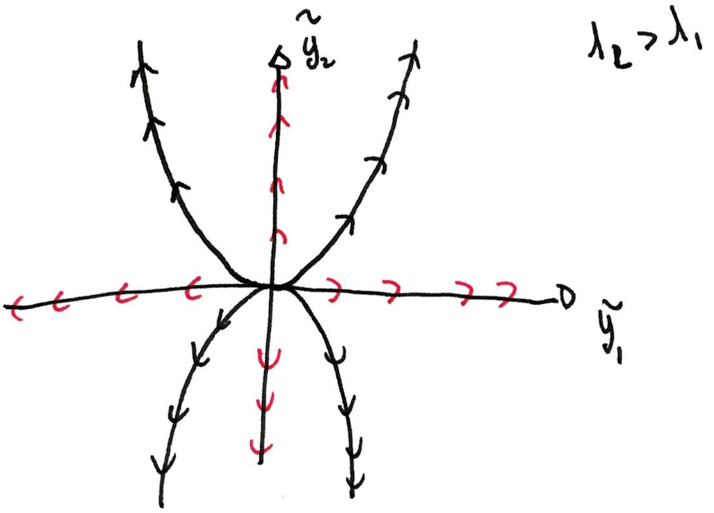
$\tilde{y}_1 \to \mathrm{sign}(D_1) \cdot \infty$

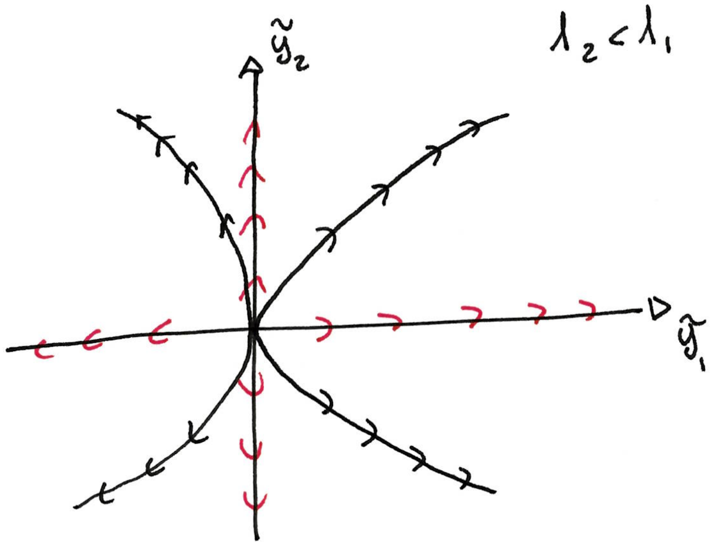
$\tilde{y}_2 \to 0$

$\boxed{2}$ Case II $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$ $\boxed{\lambda_1 > 0 \quad \lambda_2 > 0}$

Trajectories $$\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\lambda_2/\lambda_1}$$ UNSTABLE NODE

If $\frac{\lambda_2}{\lambda_1} = 2$ "$y = x^2$" these are parabolas





$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} \end{cases}$$

As $t \to \infty$

$\tilde{y}_1(t) \to \text{sign}(D_1) \cdot \infty$

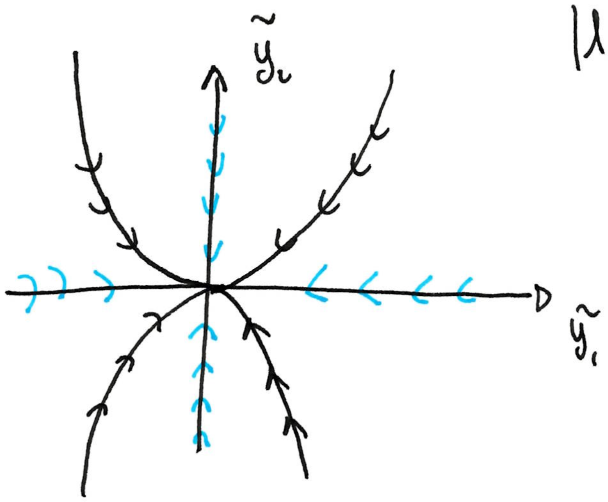
$\tilde{y}_2(t) \to \text{sign}(D_2) \cdot \infty$

$\boxed{3}$ Case III $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$ $\boxed{\lambda_1 < 0 \quad \lambda_2 < 0}$

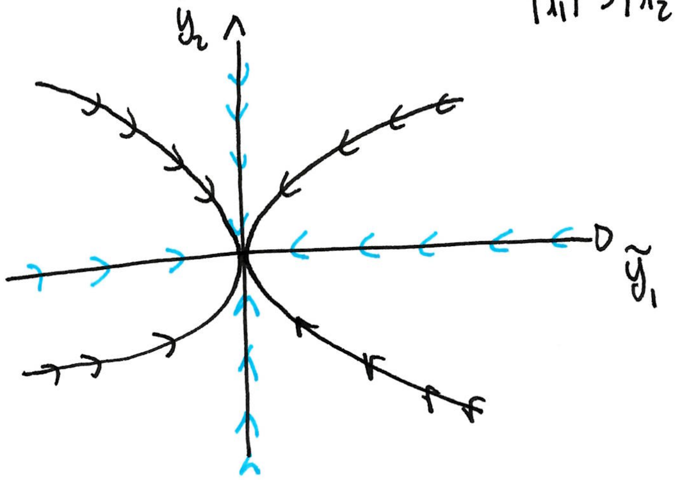
Trajectories $$\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\left|\frac{\lambda_2}{\lambda_1}\right|}$$ STABLE NODE

If $\left|\frac{\lambda_2}{\lambda_1}\right| = 2$ "$y = x^2$" these are parabolas

$|\lambda_2| > |\lambda_1|$



$|\lambda_1| > |\lambda_2|$



As $t \to \infty$

$$\begin{cases} \tilde{y}_1 = D_1 \, e^{\lambda_1 t} & (\lambda_1 < 0) & \tilde{y}_1 \to 0 \\ \tilde{y}_2 = D_2 \, e^{\lambda_2 t} & (\lambda_2 < 0) & \tilde{y}_2 \to 0 \end{cases}$$

Phase portraits: distinct and complex conjugate eigenvalues.

We will consider a linear system of ODES

$$\dot{Y} = AY$$

with A a real 2×2 matrix with complex conjugate eigenvalues

$$\boxed{\begin{aligned} \lambda_1 &= \alpha + i\beta \\ \lambda_2 &= \alpha - i\beta \end{aligned} \qquad \alpha, \beta \in \mathbb{R} \qquad \beta \neq 0}$$

We make the following considerations

1. In this case the eigenvectors u_1 and u_2 are complex conjugate

   <u>Example</u> $\quad u_1 = \begin{pmatrix} 1 \\ i\omega \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$

2. Therefore we can write

$$u_1 = v_1 - i v_2 \qquad \text{with } v_1 = \operatorname{Re} u_1$$
$$u_2 = v_1 + i v_2 \qquad v_2 = -\operatorname{Im} u_1$$

v_1 & v_2 are vectors with real components!

Example

$$u_1 = \begin{pmatrix} 1 \\ i\omega \end{pmatrix} \qquad v_1 = \mathrm{Re}\, u_1 = \mathrm{Re} \begin{pmatrix} 1 \\ i\omega \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2 = -\,\mathrm{Im}\, u_1 = -\,\mathrm{Im} \begin{pmatrix} 1 \\ i\omega \end{pmatrix} = -\begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega \end{pmatrix}$$

$$u_1 = v_1 - i v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ -\omega \end{pmatrix} = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}$$

$$u_2 = v_1 + i v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\omega \end{pmatrix} = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$$

3 v_1, v_2 are linearly independent

Therefore we can write in a unique way

$$\boxed{Y = \breve{y}_1 v_1 + \breve{y}_2 v_2}$$

$\breve{y}_1, \breve{y}_2$ define a new set of coordinates

4 The explicit solution of $\dot{Y} = AY$ reads

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

If y_1, y_2 are real, λ_1, λ_2 are complex conjugate

u_1, u_2 " " "

$$\Rightarrow \quad \boxed{D_1, D_2 \text{ are complex conjugate}}$$

We can put
$$D_1 = \frac{1}{2}\left(\tilde{a} + i\tilde{b}\right)$$
$$D_2 = \frac{1}{2}\left(\tilde{a} - i\tilde{b}\right)$$

where $\tilde{a}, \tilde{b}$ are arbitrary real constants fixed by the initial condition.