Phase portraits : Case of two distinct, real eigenvalues

Consider the linear system of ODEs

$$\dot{Y} = AY \qquad \text{where} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and A is a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with two real and ~~distinct~~ distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$
$\lambda_1 \neq \lambda_2$ and eigenvectors

$$u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \quad \text{corresponding to } \lambda_1$$

$$u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \quad \text{corresponding to } \lambda_2$$

Since the eigenvectors are *linearly independent*
there is a unique way to decompose Y into a
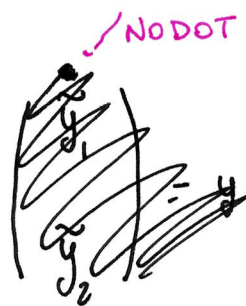linear combination of u_1 and u_2

$$\boxed{Y = \tilde{y}_1 \, u_1 + \tilde{y}_2 \, u_2}$$

Proposition   The vector $\tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$ of the new coordinates

follows the dynamical system

$$\dot{\tilde{y}} = \tilde{A}\,\tilde{y} \qquad \text{where} \quad \tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Proof   Since $Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2$ we have

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$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \tilde{y}_1 \begin{pmatrix} P_1 \\ q_1 \end{pmatrix} + \tilde{y}_2 \begin{pmatrix} P_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} \tilde{y}_1 P_1 + \tilde{y}_2 P_2 \\ \tilde{y}_1 q_1 + \tilde{y}_2 q_2 \end{pmatrix}$$

This transformation can be written in matrix form as

$$Y = U\tilde{y} \qquad \text{where} \quad U = \begin{pmatrix} P_1 & P_2 \\ q_1 & q_2 \end{pmatrix}$$

Indeed

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} P_1 \tilde{y}_1 + P_2 \tilde{y}_2 \\ q_1 \tilde{y}_1 + q_2 \tilde{y}_2 \end{pmatrix}$$

$$Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2 \iff Y = U\tilde{y}$$

- Since u_1 and u_2 are linearly independent, $\det U \neq 0$ and therefore U is INVERTIBLE

Therefore

$$U^{-1} Y = U^{-1} U \tilde{Y} \qquad \Rightarrow \boxed{\tilde{Y} = U^{-1} Y} \quad *$$

$$\underbrace{\phantom{U^{-1}U}}_{Id}$$

Differentiating with respect to time Eq. (*) we get

$$\dot{\tilde{Y}} = U^{-1} \dot{Y} = U^{-1} A Y = U^{-1} A U \tilde{Y}$$

$$\uparrow$$
$$\text{wrong}$$
$$\dot{Y} = AY$$

$$\uparrow$$
$$\text{wrong}$$
$$Y = U\tilde{Y}$$

Putting $U^{-1} A U = \tilde{A}$ we have that

$$\dot{\tilde{Y}} = \tilde{A} \tilde{Y} \qquad \text{where} \qquad \tilde{A} = U^{-1} A U$$

U is the matrix of eigenvectors of A

$\Rightarrow$ so it diagonalizes the matrix A, i.e.

$$\tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = U^{-1} A U$$

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Since $\dot{\tilde{Y}} = \tilde{A}\,\tilde{Y}$ it follows that

$$\begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \dot{\tilde{y}}_1 = \lambda_1 \, \tilde{y}_1 \\ \\ \dot{\tilde{y}}_2 = \lambda_2 \, \tilde{y}_2 \end{cases}$$

therefore $\quad \begin{cases} \tilde{y}_1 = D_1 \, e^{\lambda_1 t} \\ \\ \tilde{y}_2 = D_2 \, e^{\lambda_2 t} \end{cases} \quad$ with $\quad Y = \tilde{y}_1 \, u_1 + \tilde{y}_2 \, u_2$

In other words $\quad \tilde{y}_1(t) = c_1(t) \;$ and $\; \tilde{y}_2(t) = c_2(t) \;$ indicating

with $c_1(t)$ and $c_2(t)$ the notation used in week 9

What is the phase portrait in the new coordinates $\tilde{y}_1, \tilde{y}_2$?

Example $\qquad \dot{Y} = AY \qquad$ where $\qquad A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

with

$$\lambda_1 = 2 \qquad \Rightarrow \quad u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad \Rightarrow \quad u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

It follows that U is given by

$$U = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \det U = 1 - 2 = -1$$

$$\det U \neq 0$$

$$U^{-1} = \frac{1}{\det U} \cdot \begin{pmatrix} q_2 & -p_2 \\ -q_1 & p_1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{A} = U^{-1} A U = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \text{\textcolor{red}{[check at home]}}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

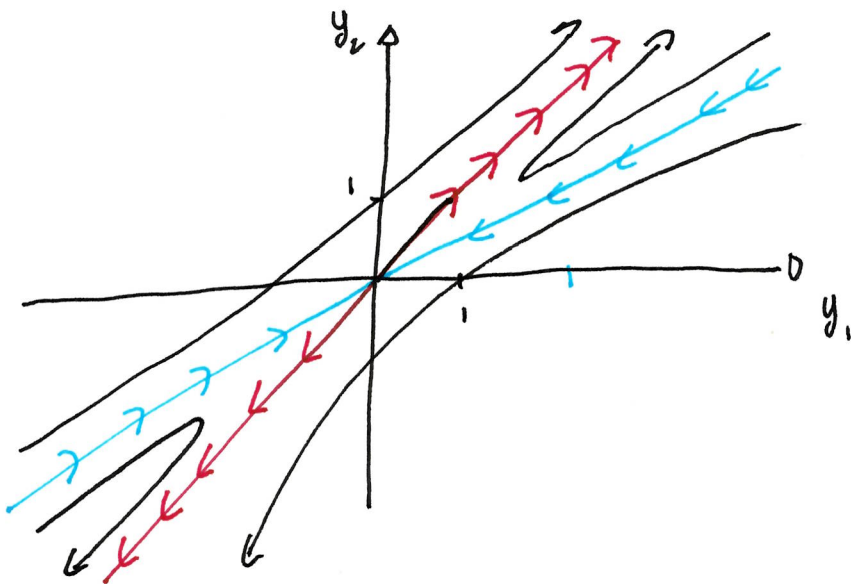We have $\quad Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2 \quad$ with $\quad \dot{\tilde{Y}} = \tilde{A} \tilde{Y}$

Therefore $\tilde{y}_1, \tilde{y}_2$ can be interpreted as new set of coordinates describing the trajectory on the basis of the eigenvectors u_1, u_2.

Phase portrait in the original and in the new coordinates
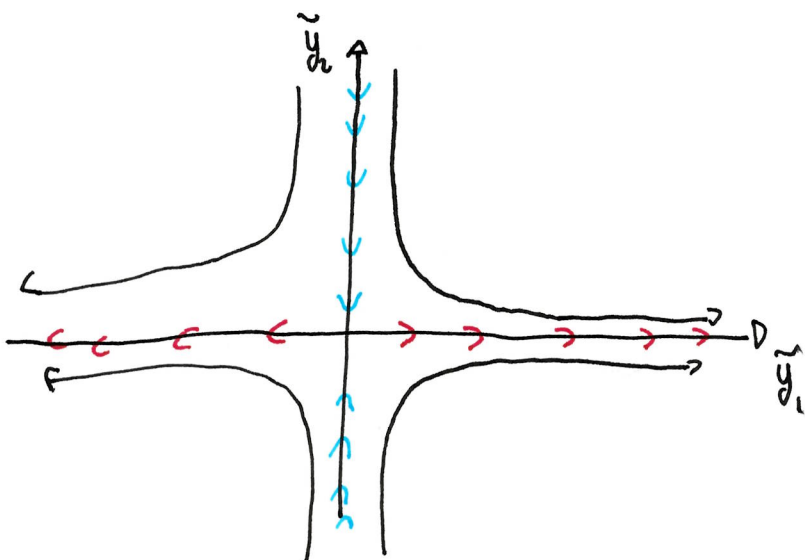
Original coordinates [example in week 9]

$\lambda_1 = 2 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1 > 0$

$\lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

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New coordinates

The phase portrait in the new coordinates is TOPOLOGICALLY EQUIVALENT to the phase portrait in the original coordinates