# MTH5123 Differential Equations

## Formative Assessment : Week 9– Solutions G. Bianconi

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### I. Tutorial Problems-Solution of selected questions

**A.** Determine the eigenvalues and eigenvectors of the following matrices:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

**Solution:** In each case, we first find the eigenvalues $\lambda$ using $\det(A = \lambda I_{2\times 2}) = 0$ and then for each $\lambda$, determine an associated eigenvector $\vec{v}$ satisfying $A\vec{v} = \lambda\vec{v}$.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = \pm i, \quad v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad \Rightarrow \quad \lambda = 2, -3, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \quad \Rightarrow \quad \lambda = 3, -1, \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

**B.** Find and sketch the solution of the initial value problems

**1)** $\dot{y_1} = -\frac{1}{2}y_1 + \frac{5}{2}y_2$ , $\dot{y_2} = \frac{5}{2}y_1 - \frac{1}{2}y_2$, $y_1(0) = a, y_2(0) = b$.

**Solution.** The matrix associated with this system is given by $A = \begin{pmatrix} -\frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{pmatrix}$. The characteristic equation is $\lambda^2 + \lambda - 6 = 0$ with two positive real roots $\lambda_1 = 2$, $\lambda_2 = -3$. The eigenvector corresponding to $\lambda_1 = 2$ can be found from

$$\begin{pmatrix} -\frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \Rightarrow q_1 = p_1$$

so that $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, for $\lambda_1 = -3$ we find $q_2 = -p_2$ so that the eigenvector is $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution has the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

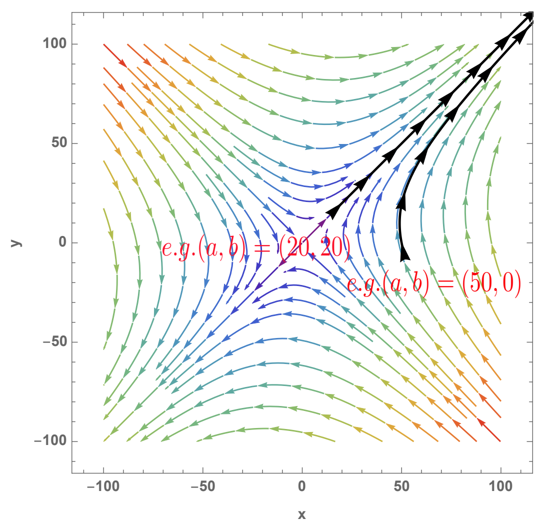
The initial conditions yield

$$y_1(0) = C_1 + C_2 = a \, , \; y_2(0) = C_1 - C_2 = b \; \Rightarrow \; C_1 = \frac{1}{2}(a+b), \; C_2 = \frac{1}{2}(a-b)$$

so that the solution to the initial value problem is given by

$$y_1 = \frac{1}{2}(a+b)e^{2t} + \frac{1}{2}(a-b)e^{-3t} \, , \; y_2 = \frac{1}{2}(a+b)e^{2t} - \frac{1}{2}(a-b)e^{-3t} \, .$$

Sketch the solution to the IVP when $y_1(0) = a, y_2(0) = b$. (This is what we practiced in CW7, where we plotted the parametric curves based on $y_1(t)$ and $y_2(t)$.) By varying the values of $(a, b)$ in the initial conditions, we have the solutions are sketched as follow,

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where we marked two example trajectories (solutions) to two different initial conditions as $y_1(0) = a = 20, y_2(0) = b = 20$, and $y_1(0) = a = 50, y_2(0) = b = 0$.

(Note: In our lecture, we explained when the initial conditions are not fixed numbers yet, we can plot solutions by assuming different values of $a$ and $b$. This will refer to different initial conditions thus different solutions in the phase plane. Once you tried enough initial conditions, you will get a rough picture of the all possible solutions, which become the phase portrait (general solution) to the ODE system. Connect this practice with our week 10 lectures.)

**2)** $\dot{y_1} = -y_1 + 5y_2 \, , \; \dot{y_2} = -y_1 + y_2 \, , \;\; y_1(0) = 0 \, , \; y_2(0) = 4.$

**Solution.** The matrix associated with this system is given by $A = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}$. The characteristic equation is $\lambda^2 + 4 = 0$ with two complex conjugate roots $\lambda_1 = 2i$, $\lambda_2 = -2i$. The eigenvector corresponding to $\lambda_1 = 2i$ can be found from

$$\begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2i \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \; \Rightarrow \; 5q_1 = (1+2i)p_1$$

so that the eigenvectors are $\mathbf{u}_1 = \begin{pmatrix} 1 \\ \frac{1}{5}(1+2i) \end{pmatrix}$ and the complex conjugate vector $\mathbf{u}_2 = \begin{pmatrix} 1 \\ \frac{1}{5}(1-2i) \end{pmatrix}$. The general solution has the form

$$\begin{pmatrix} y_1 \\ y \end{pmatrix} = C_1 e^{2it} \begin{pmatrix} 1 \\ \frac{1}{5}(1+2i) \end{pmatrix} + C_2 e^{-2it} \begin{pmatrix} 1 \\ \frac{1}{5}(1-2i) \end{pmatrix} .$$

The initial conditions yield

$$y_1(0) = C_1 + C_2 = 0\,,\; y_2(0) = C_1 \frac{1}{5}(1+2i) + C_2 \frac{1}{5}(1-2i) = 4\,, \Rightarrow\; C_1 = -5i,\; C_2 = 5i\,.$$

The solution $y_1(t)$ to the initial value problem is given by

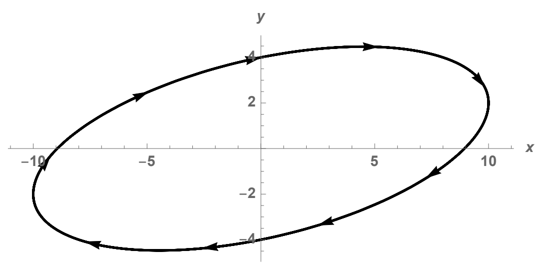
$$y_1 = -5i\left(e^{2it} - e^{-2it}\right) = 10 \sin 2t\,.$$

Similarly we have

$$y_2 = -\frac{5i}{5}(1+2i)e^{2it} + \frac{5i}{5}(1-2i)e^{-2it} = -i\left(e^{2it} - e^{-2it} + 2i\left(e^{2it} + e^{-2it}\right)\right)$$

so that

$$y_2 = 2\sin 2t + 4\cos 2t\,.$$

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**C. (1)** Linearize $\dot{y_1} = y_1 + e^{y_2} - \cos y_2$ , $\dot{y_2} = 3y_1 - y_2 - \sin y_2$ around the fixed point at $y_1 = y_2 = 0$.

**Solution.** We first linearize this system around $y_1 = y_2 = 0$.For the original nonlinear system, we can denote $f_1(y_1, y_2) = y_1 + e^{y_2} - \cos y_2$ and $f_2(y_1, y_2) = 3y_1 - y_2 - \sin y_2$. Thus, $\frac{\partial f_1(y_1,y_2)}{\partial y_1} = 1$, $\frac{\partial f_1(y_1,y_2)}{\partial y_2} = e^y + \sin y_2$, $\frac{\partial f_2(y_1,y_2)}{\partial y_1} = 3$ and $\frac{\partial f_1(y_1,y_2)}{\partial y_2} = -1 - \cos y_2$.

At the fixed point (or called as equilibrium) $(0,0)$, $\frac{\partial f_1(y_1,y_2)}{\partial y_1}|_{(0,0)} = 1$, $\frac{\partial f_1(y_1,y_2)}{\partial y_2}|_{(0,0)} = 1 + \sin 0 = 1$. Because $\frac{\partial f_2(y_1,y_2)}{\partial y_1} = 3$ and $\frac{\partial f_1(y_1,y_2)}{\partial y_2} = -1 - 1 = -2$. Thus, system can be linearised as

$$\dot{y_1} = y_1 + y_2\,,\; \dot{y_2} = 3y_1 - 2y_2\,.$$

The matrix associated with the linearized system is given by $A = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$. The characteristic equation is $(1-\lambda)(-2-\lambda) - 3 = \lambda^2 + \lambda - 5 = 0$ with two real roots $\lambda_1 = \frac{-1+\sqrt{21}}{2} > 0$,

$\lambda_2 = \frac{-1-\sqrt{21}}{2} < 0$.

**(2)** Linearize the following equation $\dot{y_1} = -2y_1 - 3y_2 + y_1^5$ , $\dot{y_2} = y_1 + y_2 - y_2^2$ around the fixed point at $y_1 = y_2 = 0$ and find the eigenvalues.

**Solution.** The matrix associated with the linearized system is given by $A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$. The characteristic equation is $\lambda^2 + \lambda + 1 = 0$ with two complex conjugate roots $\lambda_1 = \frac{-1+i\sqrt{3}}{2}$, $\lambda_2 = \frac{-1-i\sqrt{3}}{2}$.

**D.** (1) Compute all equilibria of the non-linear ODE system

$$\dot{y_1} = -y_1 + 3y_2 - y_1^2 + 3y_1y_2 \; , \; \dot{y_2} = -3y_1 - y_2 \; ,$$

and linearise this ODE systems around its equilibria separately and write down their matrix forms.

**Solution.** The non-linear system of ODE can be written as

$$\dot{y_1} = (-y_1 + 3y_2)(1 + y_1) \; , \; \dot{y_2} = -3y_1 - y_2 \; ,$$

. The equilibrium points are $(y_1, y_2) = (0, 0)$ and $(y_1, y_2) = (-1, 3)$. The matrix associated with the linearized system around the equilibrium point $(y_1, y_2) = (0, 0)$ is given by $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

The matrix associated with the linearized system around the equilibrium point $(y_1, y_2) = (-1, 3)$ is given by $A = \begin{pmatrix} 10 & 0 \\ -3 & -1 \end{pmatrix}$.

(2) Determine the general solution of the linearised system at $\mathbf{y} = 0$. Find the function $x(t)$ that solves the initial value problem for the system above specified by the initial conditions

$$y_1(0) = a \; , \; y_2(0) = b$$

and express it in terms of real-valued functions. Sketch the trajectories of this autonomous system in phase space when $a = b = 1$.

**Solution.** The matrix $A \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 10 = 0$ and eigenvalues $\lambda = -1 \pm 3i$ therefore we expect the phase portrait to be a stable focus. The eigenvector corresponding to the eigenvalue $\lambda = -1 + 3i$ is $\mathbf{u}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. The eigenvector corresponding to the eigenvalue $\lambda = -1 - 3i$ is $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The general solution of this dynamical system of ODEs is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{(-1+3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-(1+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} .$$

where $C_2$ must be complex conjugate of $C_1$ to guarantee that $y_1$ and $y_2$ are both real. Imposing the initial condition we obtain $2\text{Re}(C_1) = a$ and $-2\text{Im}(C_1) = b$, hence $C_1 = \frac{a}{2} - i\frac{b}{2}$. Therefore we obtain for $y_1 = y_1(t)$

$$y_1 = 2\text{Re}\left[C_1 e^{(-1+3i)t}\right] = e^{-t}(a\cos(3t) + b\sin(3t))$$

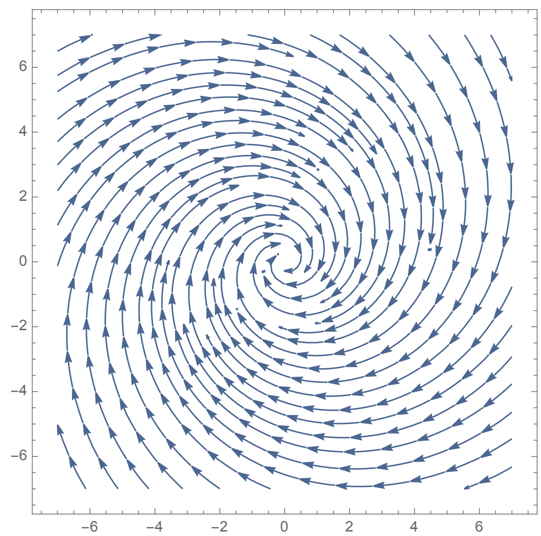
and for $y_2 = y_2(t)$

$$y_2 = -2\text{Im}\left[C_1 e^{(-1+3i)t}\right] = e^{-t}(-a\sin(3t) + b\cos(3t))$$

. This trajectory is a stable focus and we have $y_1^2 + y_2^2 = e^{-2t}(a^2 + b^2)$. Assume $a = 1$ and $b = 0$. Then we have

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} == e^{-t} \begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix}.$$

The solution at time $t = 0$ is $\mathbf{y}(0) = (1, 0)$, with $\dot{\mathbf{y}}(0) = (0, -3)$. Therefore the direction of the spiral is clockwise.



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## III. Graphing trajectories and analysis of dynamical systems

**A.** Consider the dynamical system given by $\begin{cases} \dot{y_1} &= y_2 \\ \dot{y_2} &= -2y_1 + 2y_2 \end{cases}$.

**1)** Rewrite the system in matrix form and find the eigenvalues and eigenvectors of the associated coefficient matrix.

**Solution.** The system is written as

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Eigenvalues and eigenvectors of the matrix are $\lambda = 1 \pm i$ and $v = (1 \mp i, 2)$, respectively.

**2)** Find the general solution of the system of ODEs, justifying your answer.

**Solution.** Note that the general solution of the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = C_1 e^{(1+i)t} \begin{bmatrix} 1-i \\ 2 \end{bmatrix} + C_2 e^{(1-i)t} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

gives complex curves. To find the associated *real solutions*, use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$, and assume $\bar{C}_1 = C_2$ (*WHY?*) to solve for the (real) initial conditions $y_1(0) = 0$, $y_2(0) = 1$. You should get $C_1 = (1-i)/4$. Then, sketch the trajectory in the $x$-$y$ plane for part **3)**, which should be an outward-pointing clockwise skew spiral (*use a computer to verify this!*).