The matrix approach to linear system of ODEs

We consider the linear system of ODES

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (1)$$

where A is the matrix $\qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Indicating $\quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we can write (1)

in matrix form as $\qquad \dot{Y} = A Y \qquad (2)$

In this week and in the next we will use algebra

to solve the linear system of ODEs (Eq (1) or equivalently Eq. (2))

Consider the linear system

$$\dot{Y} = AY$$

with A having two distinct eigenvalues $\lambda_1 \neq \lambda_2$

Let us show that the general solution to this linear system of ODEs can be written as.

$$\boxed{Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2} \quad (*)$$

where D_1, D_2 are arbitrary constants.

where $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$ are the eigenvectors of A satisfying

$$A u_1 = \lambda_1 u_1$$
$$A u_2 = \lambda_2 u_2$$

Eq (*) can be written in matrix form as

$$\boxed{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = D_1 e^{\lambda_1 t} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + D_2 e^{\lambda_2 t} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}}$$

Derivation

Given that A has two distinct eigenvalues $\lambda_1 \neq \lambda_2$ then the corresponding eigenvectors u_1 and u_2 are <u>linearly independent</u>

- This means that any 2-dimensional vector can be written as a linear combination of u_1 and u_2 in a <u>unique way</u>

- It follows that $Y(t)$ can be expressed in a unique way as

$$\boxed{Y(t) = c_1(t)\, u_1 + c_2(t)\, u_2} \qquad (3)$$

Now since $\underset{(2)}{\dot{Y} = AY}$ we substitute (3) into (2) we get

LHS: $\dot{Y} = \dot{c}_1(t)\, u_1 + \dot{c}_2(t)\, u_2$

RHS: $AY = A\left[c_1(t) u_1 + c_2(t)\, u_2\right] = c_1(t) \underbrace{Au_1}_{Au_1 = \lambda_1 u_1} + c_2(t) \overbrace{Au_2}^{Au_2 = \lambda_2 u_2}$

$$AY = c_1(t)\, \lambda_1\, u_1 + c_2(t)\, \lambda_2\, u_2$$

Therefore $\dot{Y} = AY$

$$\dot{c}_1(t)\, u_1 + \dot{c}_2(t)\, u_2 = c_1(t)\, \lambda_1 u_1 + c_2(t)\, \lambda_2 u_2$$

Rearranging the terms

$$\underbrace{\left[\dot{c}_1(t) - \lambda_1 c_1(t)\right]}_{=0} u_1 + \underbrace{\left[\dot{c}_2(t) - \lambda_2 c_2(t)\right]}_{=0} u_2 = 0$$

Since the eigenvectors u_1 and u_2 are independent this equation implies

$$\begin{cases} \dot{c}_1(t) - \lambda_1 c_1(t) = 0 \\ \dot{c}_2(t) - \lambda_2 c_2(t) = 0 \end{cases} \Rightarrow \begin{cases} \dot{c}_1(t) = \lambda_1 c_1(t) \\ \dot{c}_2(t) = \lambda_2 c_2(t) \end{cases}$$

$$\dot{c}_1 = \lambda_1 c_1 \qquad \text{"}\dot{y} = \lambda y\text{"}$$

Solve by separation of variables

$$\int \frac{dc_1}{c_1} = \int \lambda_1\, dt + C' \Rightarrow \ln|c_1| = \lambda_1 t + C'$$

$$\Rightarrow \boxed{c_1(t) = D_1 e^{\lambda_1 t}}$$

where D_1 is an arbitrary constant.

Therefore we set
$$\begin{cases} c_1(t) = D_1 e^{\lambda_1 t} \\ c_2(t) = D_2 e^{\lambda_2 t} \end{cases}$$

Since $Y(t) = c_1(t) u_1 + c_2(t) u_2$ we set

$$\boxed{Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2}$$

where D_1, D_2 are arbitrary constants.

General solution to $\dot{Y} = AY$

<u>Example</u> Find the general solution of $\dot{Y} = AY$ with

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

This implies
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

<u>Solution</u> A has eigenvalues and eigenvectors given by

$$\lambda_1 = 2 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_1 \neq \lambda_2$$

$$\lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The general solution is

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

$$Y(t) = D_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} D_1 e^{2t} + D_2 e^{-t} \cdot 2 \\ D_1 e^{2t} + D_2 e^{-t} \end{pmatrix}$$

$$\begin{cases} y_1(t) = D_1 e^{2t} + 2 D_2 e^{-t} \\ y_2(t) = D_1 e^{2t} + D_2 e^{-t} \end{cases}$$

General solution to

$\dot{Y} = AY$

Provide the solution to the IVP

$\dot{Y} = AY$ with I.C. $Y(0) = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where $k \in \mathbb{R}$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = k u_1$$

The general solution is $Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$

Imposing the I.C. $Y(0) = D_1 u_1 + D_2 u_2 = k u_1$

Rearranging: $\underbrace{[D_1 - k]}_{=0} u_1 + \underbrace{D_2}_{=0} u_2 = 0$

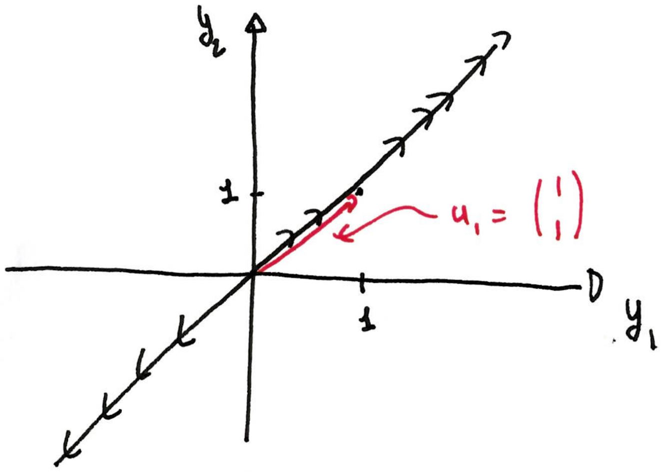
$$\Rightarrow \begin{cases} D_1 = k \\ D_2 = 0 \end{cases} \Rightarrow$$

The solution to the IVP

$$\boxed{Y(t) = k e^{\lambda_1 t} u_1 = k e^{2t} u_1 = k e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The solution to the IVP is

$$Y(t) = k e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = k e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



The trajectories with I.C. $Y(0) = k\, u_1$ move along the direction of the eigenvector u_1

u_1 is called INVARIANT MANIFOLD

If $k > 0$

$$\begin{cases} y_1(t) = k e^{2t} \\ y_2(t) = k e^{2t} \end{cases} \qquad \frac{y_1}{y_2} = 1$$

If $t \to \infty$ $\quad y_1(t) \to +\infty \quad y_2(t) \to \infty$

If $k < 0$

$$\begin{cases} y_1(t) = k e^{2t} \\ y_2(t) = k e^{2t} \end{cases} \qquad \frac{y_1}{y_2} = 1$$

If $t \to \infty$ $\quad y_1(t) \to -\infty \quad y_2 \to -\infty$

The invariant manifold is UNSTABLE

The trajectories move away from the origin.