A review of algebra

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- the <u>trace of A</u> indicated as $\mathrm{Tr}\,A$ is given by

$$\mathrm{Tr}\,A = a_{11} + a_{22} \qquad \text{(sum of diagonal terms)}$$

- the <u>determinant of a 2x2 matrix A</u> is given by

$$\det A = a_{11} a_{22} - a_{12} a_{21}$$

If the $\det A \neq 0$ we can define the <u>inverse of A</u> indicated as A^{-1} and satisfying

$$A A^{-1} = A^{-1} A = Id \qquad \text{where } Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A^{-1} can be expressed as

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

If A^{-1} exists the equation

$$AY = b$$

has solution $\quad Y = A^{-1} b$

Indeed $\quad A^{-1} A Y = A^{-1} b \Rightarrow Y = A^{-1} b \quad \square$

Eigenvalues and eigenvectors of A

Let $\quad u = \begin{pmatrix} p \\ q \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

if $\quad Au = \lambda u \qquad \lambda \in \mathbb{R}$

λ is called the eigenvalue of A

u is called the eigenvector of A corresponding to the eigenvalue λ.

1 Theorem For any 2x2 matrix A there are two eigenvalues which are the roots of the quadratic equation

$$\det(A - \lambda \, Id) = 0 \qquad (3)$$

or equivalently

$$\lambda^2 - (Tr A)\lambda + \det A = 0 \qquad (4)$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(A - \lambda Id) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$$

$$\lambda^2 - (\mathrm{Tr}\, A)\lambda + \det A = 0 \qquad \text{[check at home]}$$

It follows that there are 3 possibilities:

a) λ_1, λ_2 are distinct and real roots $\quad \lambda_1 \neq \lambda_2, \; \lambda_1, \lambda_2 \in \mathbb{R}$

b) $\lambda_1 = \lambda_2$ are identical real roots $\quad \lambda_1 = \lambda_2, \; \lambda_1, \lambda_2 \in \mathbb{R}$

c) λ_1, λ_2 are complex conjugate roots

$$\lambda_1 = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R} \quad \beta \neq 0$$
$$\lambda_2 = \alpha - i\beta$$

② If $\lambda_1 \neq \lambda_2$ (cases (a) and (c)) the two eigenvectors u_1 and u_2, each one determined up to a constant factor, are <u>linearly independent</u>

This means that $u_2 \neq k u_1$ with $k \in \mathbb{R}$

If u_1 and u_2 are linearly independent, any 2-dimensional vector y can be written <u>in a unique way</u> as

$$y = c_1 u_1 + c_2 u_2 \qquad \text{where } c_1, c_2 \in \mathbb{R}$$

Therefore if $y = c_1 u_1 + c_2 u_2 = 0$ $*$ $\Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

Assuming $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$, Eq $(*)$

can also be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + c_2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

③ If $a_{12} = a_{21}$ then either

a) $\lambda_1 = \lambda_2$

or

b) the eigenvectors u_1 and u_2 are <u>orthogonal</u>

i.e. $u_1 u_2^T = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} (p_2 \; q_2) = p_1 p_2 + q_1 q_2 = 0$

Example of calculation of eigenvalues and eigenvectors of a matrix

Calculate the eigenvalues and the eigenvector of the matrix A

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

<u>Eigenvalues</u> To find the eigenvalues we impose

$$\det(A - \lambda Id) = 0$$

$$\det(A - \lambda Id) = \det \begin{pmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{pmatrix} = 0$$

$$(-4-\lambda)(5-\lambda) - 6(-3) = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$$

$\lambda^2 - \lambda - 2 = 0$

Let us find the roots λ_1, λ_2

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases} \qquad \boxed{\lambda_1 = 2} \quad \boxed{\lambda_2 = -1}$$

$\lambda_1 \neq \lambda_2 \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1, \lambda_2$ are real and distinct.

<u>Eigenvectors</u> For eigenvalue $\lambda_1 = 2$ we will have the eigenvector $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ satisfying $Au_1 = \lambda_1 u_1$, ie.

$$\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

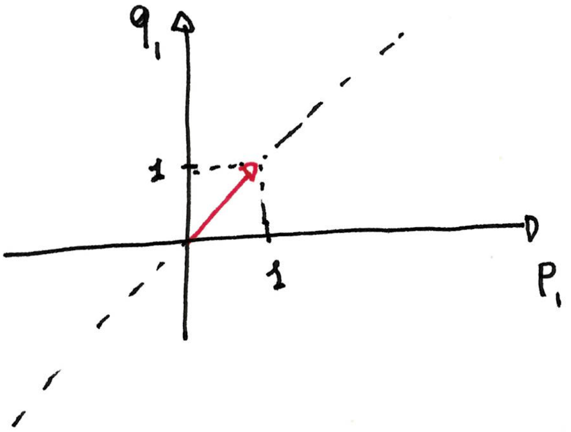
$$\begin{cases} -4p_1 + 6q_1 = \lambda_1 p_1 = 2p_1 \\ -3p_1 + 5q_1 = \lambda_1 q_1 = 2q_1 \end{cases}$$

$$\begin{cases} -\cancel{6}p_1 + \cancel{6}q_1 = 0 \\ -\cancel{3}p + \cancel{3}q_1 = 0 \end{cases} \qquad \begin{cases} p_1 = q_1 \\ p_1 = q_1 \end{cases}$$

If $p_1 = 1 \Rightarrow q_1 = 1$ $\qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvectors are always determined up to a non-zero factor

We have chosen $p_1 = 1 \Rightarrow q_1 = p_1 = 1 \Rightarrow q_1 = 1$



With a similar procedure you can find the eigenvalue

$u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_2 = -1$

$u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or any vector differing by a constant factor.