Linear and non-linear autonomous systems of 1^{st}-order ODES

A linear autonomous system of 1^{st}-order ODEs with two dependent variables y_1, y_2 reads

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (1)$$

where A is a 2×2 matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{with } a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$$

This is equivalent to the system

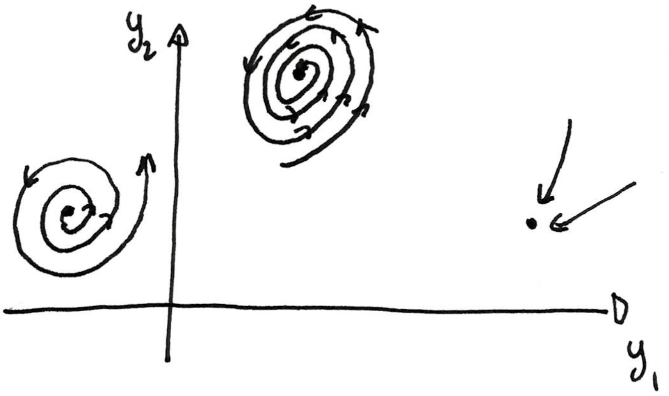
$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases} \qquad (1')$$

A autonomous dynamical system that is not in the form (1) or (1')

is called NON-LINEAR.

Non linear dynamical systems and their linearisation

Consider a autonomous non-linear system with many equilibria.



In this example we have

3 equilibria

Which are the typical trajectories close to these equilibria?

We can study these trajectories by linearisation (weeks 8-11)

We will consider ISOLATED equilibria only.

For each equilibrium point (y_1^*, y_2^*) there is a $R > 0$ such that inside the circle centered at the equilibrium point and with radius R there are no other equilibria.

Revision of Taylor series (calculus)

(1) Consider a function $f(y)$ that is infinitely differentiable at point $y = y^*$. The function can be expressed as the Taylor power series

$$\underbrace{f(y) = f(y^*) + \frac{1}{1!} f'(y^*)(y-y^*)}_{\text{linear approximation}} + \frac{1}{2!} f''(y^*)(y-y^*)^2 + \cdots$$

(2) Consider a two variable function $f(y_1, y_2)$ that is infinitely differentiable at (y_1^*, y_2^*). The function can be expressed as the Taylor power series.

$$f(y_1, y_2) = \underbrace{f(y_1^*, y_2^*) + \left.\frac{\partial f}{\partial y_1}\right|_{(y_1^*, y_2^*) = (y_1, y_2)} (y_1 - y_1^*) + \left.\frac{\partial f}{\partial y_2}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_2 - y_2^*)}_{\text{linear approximation}} + \cdots$$

Linearisation of a non-linear autonomous system around the equilibrium point (y_1^*, y_2^*)

We consider the non-linear autonomous system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

We assume that (y_1^*, y_2^*) is an equilibrium point.

So, by definition we have

$$f_1(y_1^*, y_2^*) = f_2(y_1^*, y_2^*) = 0$$

We consider (y_1, y_2) close to the fixed point (y_1^*, y_2^*)

we linearise $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$

$$f_1(y_1, y_2) = \overbrace{f_1(y_1^*, y_2^*)}^{=0} + \overbrace{\left.\frac{\partial f_1}{\partial y_1}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}}^{=a_{11}} (y_1 - y_1^*) + \overbrace{\left.\frac{\partial f_1}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}}^{=a_{12}} (y_2 - y_2^*)$$

$$f_2(y_1, y_2) = \overbrace{f_2(y_1^*, y_2^*)}^{=0} + \overbrace{\left.\frac{\partial f_2}{\partial y_1}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}}^{=a_{21}} (y_1 - y_1^*) + \overbrace{\left.\frac{\partial f_2}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}}^{=a_{22}} (y_2 - y_2^*)$$

We truncate the expansion to the linear order.

By setting

$$A = \begin{pmatrix} \overset{=a_{11}}{\left.\dfrac{\partial f_1}{\partial y_1}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} & \overset{=a_{12}}{\left.\dfrac{\partial f_1}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} \\ \overset{=a_{21}}{\left.\dfrac{\partial f_2}{\partial y_1}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} & \overset{=a_{22}}{\left.\dfrac{\partial f_2}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} \end{pmatrix}$$

we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We obtain in this way

$$f_1(y_1,y_2) = a_{11}\,(y_1 - y_1^*) + a_{12}\,(y_2 - y_2^*)$$

$$f_2(y_1,y_2) = a_{21}\,(y_1 - y_1^*) + a_{22}\,(y_2 - y_2^*)$$

or equivalently

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 - y_1^* \\ y_2 - y_2^* \end{pmatrix}$$

We can make the change of variables

$$\begin{cases} \tilde{y}_1 = y_1 - y_1^* \\ \tilde{y}_2 = y_2 - y_2^* \end{cases} \Longrightarrow \begin{cases} \dot{\tilde{y}}_1 = \dot{y}_1 \\ \dot{\tilde{y}}_2 = \dot{y}_2 \end{cases}$$

Therefore we obtain

$$\begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = A \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$$

This is the linearised system close to the equilibrium point $(y_1, y_2) = (y_1^*, y_2^*)$

$\boxed{\text{The dynamical behavior of the linearised system around the equilibrium point is much easier to study than the original non-linear system!}}$

<u>Example</u> Linearise the system of ODEs close to the equilibrium point $(0,0)$.

$$\begin{cases} \dot{y}_1 = \sin y_1 + e^{y_2} - 1 & = f_1(y_1, y_2) \\ \dot{y}_2 = 3 \sin y_1 & = f_2(y_1, y_2) \end{cases}$$

(1) Check $(0,0)$ is an equilibrium point, i.e. $f_1(0,0) = f_2(0,0) = 0$

$f_1(0,0) = \sin 0 + e^0 - 1 = 0$ ✓

$f_2(0,0) = 3 \sin 0 = 0$ ✓

$(0,0)$ is an equilibrium point

② Let us linearise the non linear system around the point

$(0,0) = (y_1^*, y_2^*)$.

The linearised system of ODEs reads $\tilde{y}_1 = y_1 \quad \tilde{y}_2 = y_2$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} \left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_1}{\partial y_2}\right|_{0,0} \\ \left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_2}{\partial y_2}\right|_{0,0} \end{pmatrix}$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = \left.\frac{\partial}{\partial y_1}\left(\sin y_1 + e^{y_2} - 1\right)\right|_{(0,0)} = \left.\cos y_1\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = \left.\frac{\partial}{\partial y_2}\left(\sin y_1 + e^{y_2} - 1\right)\right|_{(0,0)} = \left.e^{y_2}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = \left.\frac{\partial}{\partial y_1} 3(\sin y_1)\right|_{(0,0)} = \left.3\cos y_1\right|_{(0,0)} = 3$$

$$\left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \left.\frac{\partial}{\partial y_2} 3(\sin y_1)\right|_{(0,0)} = 0$$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

Linearised system!