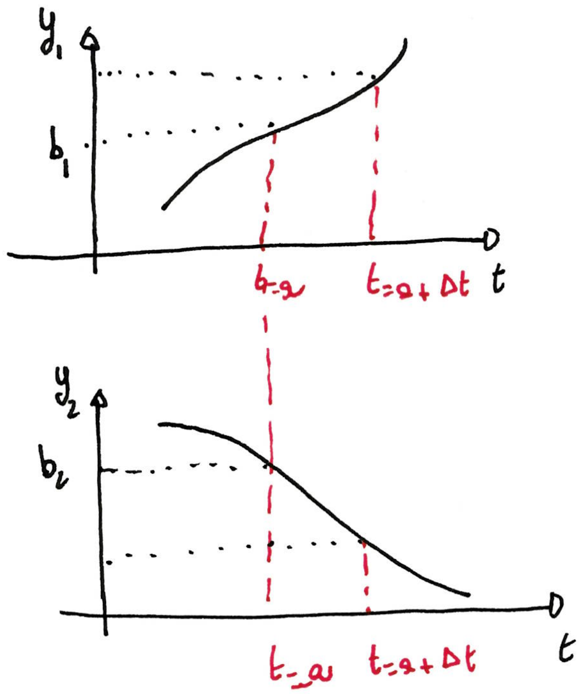
Trajectories of dynamical systems

Let us consider an autonomous dynamical system

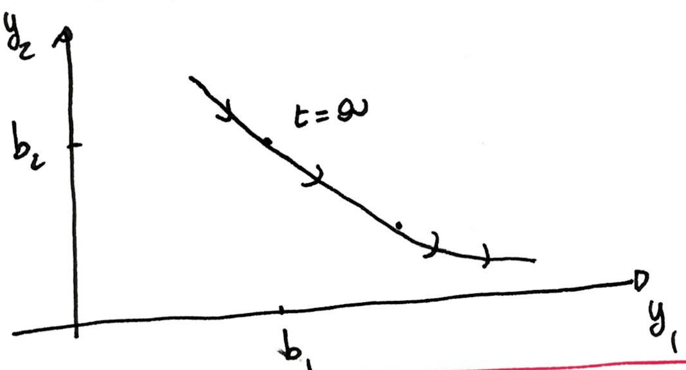
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \quad \text{and} \quad \text{ICs: } \begin{cases} y_1(a) = b_1 \\ y_2(a) = b_2 \end{cases}$$

satisfying the hypothesis of the Picard-Lindelöf theorem

and let us consider the solution $\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \end{cases}$



TRAJECTORY



The solution $y_1 = y_1(t)$ and $y_2 = y_2(t)$ describes a parametrised curve in the phase space with parameter t

This is the TRAJECTORY

Example

$$\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = 2y_2 \end{cases} \qquad \text{ICs:} \begin{cases} y_1(0) = 1 \\ y_2(0) = 1 \end{cases}$$

Solving $\dot{y}_1 = y_1$ & $y_1(0) = 1$

[check at home] $\boxed{y_1 = e^t}$

Solving $\dot{y}_2 = 2y_2$ & $y_2(0) = 1$

[check at home] $\boxed{y_2 = e^{2t}}$

Solution of the IVP of the ~~[illegible]~~ dynamical system

$$\begin{cases} y_1 = e^t \\ y_2 = e^{2t} \end{cases}$$

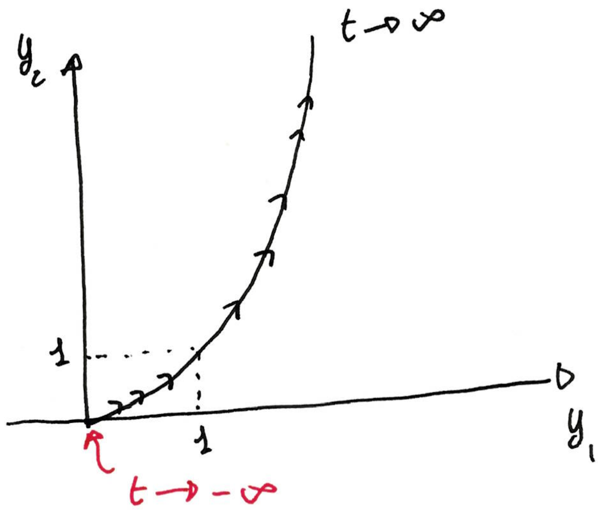
$y_1 = e^t \qquad y_1 > 0$

$y_2 = e^{2t} = (e^t)^2 = y_1^2 \qquad y_2 > 0$

Trajectory

$\boxed{y_2 = y_1^2}$

The trajectory is a parabola!



Trajectories and their properties

For a dynamical system satisfying the hypothesis of the Picard-Lindelöf theorem, trajectories either completely coincide or they do not have any point in common

If two trajectories have a point in common, then using that point as an initial condition, we would find two different solutions to the IVP.
This however contradict the Picard-Lindelöf theorem.
So trajectories cannot cross.

Equilibria of dynamical systems

The autonomous system

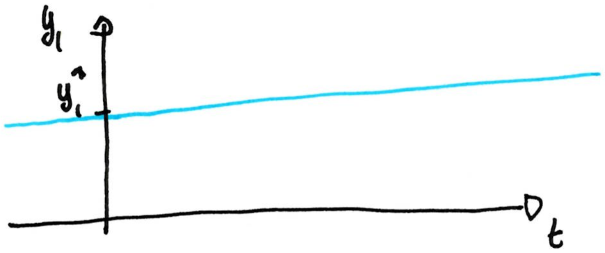
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

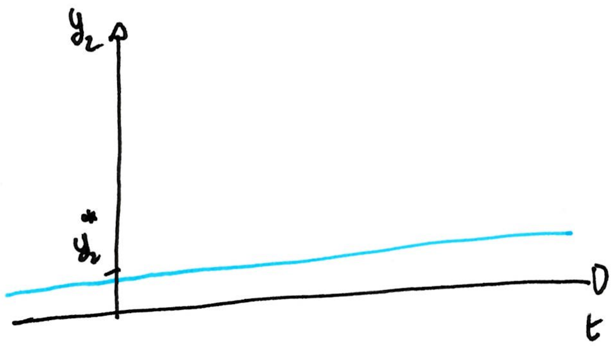
has an EQUILIBRIUM POINT at $(y_1, y_2) = (y_1^*, y_2^*)$ if and only if

$$\begin{cases} \dot{y}_1 = f_1(y_1^*, y_2^*) = 0 \\ \dot{y}_2 = f_2(y_1^*, y_2^*) = 0 \end{cases}$$

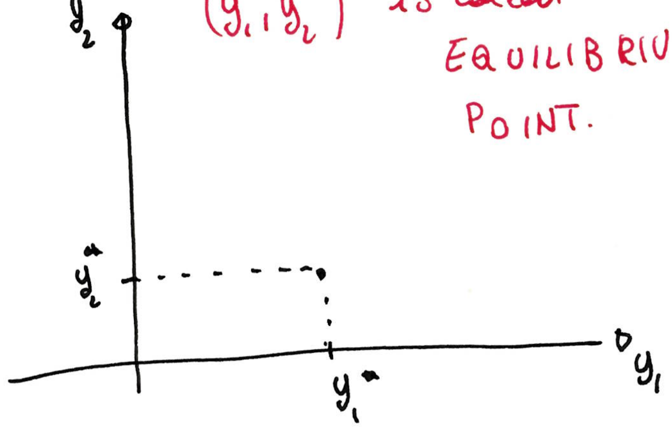
From this definition it follows that IVP with

ICs: $\begin{cases} y_1(a) = y_1^* \\ y_2(a) = y_2^* \end{cases}$ has solution $\begin{cases} y_1(t) = y_1^* \\ y_2(t) = y_2^* \end{cases}$





(y_1^*, y_2^*) is called EQUILIBRIUM POINT.



The trajectory of the equilibrium point (y_1^*, y_2^*)

Equilibria

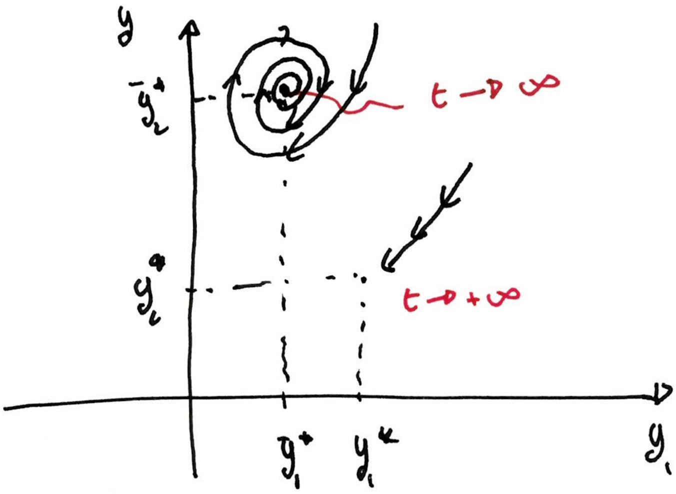
A dynamical system can have many equilibria

[Equilibrium - singular
[Equilibria - plural

Equilibria are also called FIXED POINTS, STATIONARY POINTS or SINGULAR POINTS

Under the hypothesis of the Picard-Lindelöf theorem any solution to an IVP with ICs different from an equilibrium point can only reach an equilibrium point ASYMPTOTICALLY IN TIME, i.e. for $t \to \infty$ or $t \to -\infty$. Cannot reach an equilibrium point at finite time

An equilibrium point is a trajectory
Trajectories do not cross.
It follows that an equilibrium point can only be reached for $t \to \infty$ or $t \to -\infty$



Finding equilibria

Example Find all the equilibria of the following dynamical system

$$\begin{cases} \dot{y}_1 = 4 y_1 y_2 - 2 & = f_1(y_1, y_2) \\ \dot{y}_2 = (y_1 - 2)(y_2 - 2y_1) = f_2(y_1, y_2) \end{cases}$$

Solution: At the equilibrium point $(y_1, y_2) = (y_1^*, y_2^*)$ we must have $\dot{y}_1 = \dot{y}_2 = 0$

So we impose

$$\begin{cases} 0 = f_1(y_1, y_2) = 4 y_1 y_2 - 2 \\ 0 = f_2(y_1, y_2) = (y_1 - 2)(y_2 - 2y_1) \end{cases}$$

From $4 y_1 y_2 - 2 = 0 \Longrightarrow y_1 y_2 = \frac{1}{2}$

From $(y_1 - 2)(y_2 - 2y_1) = 0 \Longrightarrow$ Either $y_1 = 2$

or $y_2 = 2y_1$

- If $y_1 = 2$ $\quad y_1 y_2 = \frac{1}{2}$ $\quad \Rightarrow y_2 = \frac{1}{4}$

$$(y_1^*, y_2^*) = \left(2, \frac{1}{4}\right)$$ 1^{st}-equilibrium points

- If $y_2 = 2y_1$ $\quad y_1 y_2 = \frac{1}{2}$ $\quad \Rightarrow 2y_1^2 = \frac{1}{2}$ $\quad \Rightarrow y_1^2 = \frac{1}{4}$

$$\Rightarrow y_1 = \pm \frac{1}{2}$$

$$\Rightarrow (y_1^*, y_2^*) = \left(\frac{1}{2}, 1\right)$$ 2^{nd}-equilibrium point

$$\Rightarrow (y_1^*, y_2^*) = \left(-\frac{1}{2}, -1\right)$$ 3^{rd}-equilibrium point