Wellecome back to MTH5123 Differential Equations!

Module

Weeks 1-3	1^{st}-order ODEs, IVP, Picard-Lindelöf theorem
Weeks 4-6	2^{nd}-order ODEs, BVP, Theorem of the Alternative
Weeks 8-11	Systems of ODEs of 1^{st}-order, Phase portraits
Week 12	Revision week

Assessment

Coursework 1 — 10% final mark

Well done! — Average mark 73%

Std deviation 15%

Coursework 2 — 10% final mark

Friday 8 Dec - 15 Dec

Covering material weeks 8-11

Final exam

80% Final mark

4 Questions

Questions 1-3 Week 1-6

Question 4 Week 8-11

Handwritten - No calculator

Appendix is on QM+ Tab Week 7

Train with :
- Formative Ass.
- Mock Quizzes
- Past papers. (Week 7 Tab on QM+)

System of 1st-order ODEs

A system of 1st-order ODEs having t as independent variable and $y_1, y_2 \dots y_n$ as dependent variable reads in normal form

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2, \dots y_n) \\ \dot{y}_2 = f_2(t, y_1, y_2, \dots y_n) \\ \vdots \\ \dot{y}_m = f_m(t, y_1, y_2, \dots y_m) \end{cases} \qquad (1)$$

A system of ODEs having t as independent variable, such as (1), is called DYNAMICAL SYSTEM.

A dynamical system is AUTONOMOUS if all the functions $f_i(t, y_1, y_2 \dots y_m)$ are independent of time

$$f_i(t, y_1, y_2 \dots y_n) = f_i(y_1, y_2 \dots y_n)$$

otherwise the dynamical system is called NON-AUTONOMOUS.

In this module we will only consider dynamical systems of 1^{st}-order ODEs with two dependent variables

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2) \\ \dot{y}_2 = f_2(t, y_1, y_2) \end{cases} \quad \text{or} \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix}$$

Examples

$$\begin{cases} \dot{y}_1 = y_1 + e^{y_2} = f_1(y_1, y_2) \\ \dot{y}_2 = (y_1 + y_2)^{1/3} = f_2(y_1, y_2) \end{cases}$$

AUTONOMOUS

$$\begin{cases} \dot{y}_1 = e^{t} + y_1 y_3^2 \\ \dot{y}_2 = \sin y_1 \end{cases}$$

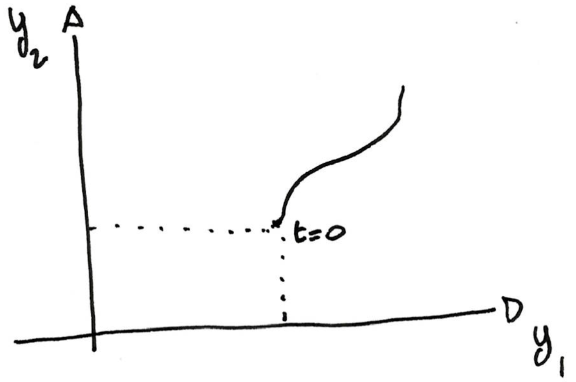
NON-AUTONOMOUS

Phase space

Consider the autonomous dynamical system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

A region of the two dimensional space $\mathbb{R}^2$ ~~is~~ described by the coordinates (y_1, y_2) where the functions $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are well defined is called the **PHASE SPACE** of the system.



We will consider only dynamical systems where the phase-space is the (y_1, y_2) $\mathbb{R}^2$ plane.

Initial Value Proplem for Autonomous Dynamical Systems

The IVP for autonomous dynamical system comprises

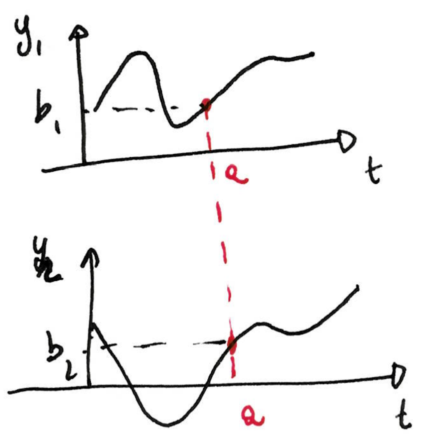
System of ODEs:
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

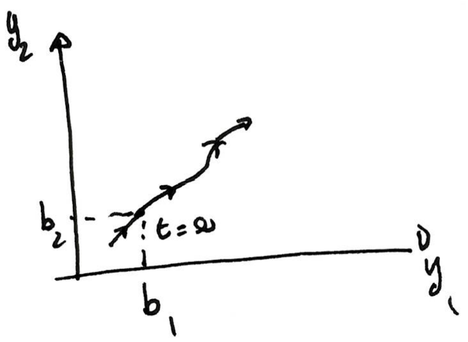
ICs:
$$y_1(a) = b_1 \qquad \text{at time } t = a$$
$$y_2(a) = b_2$$

We will consider solutions with $t \in (-\infty, \infty)$ including both future and past.

In the hypothesis in which the functions f_1, f_2 and their partial derivatives $\frac{\partial f_i}{\partial y_j}$ $i \in \{1,2\}, j \in \{1,2\}$ are continous in $\mathbb{R}^2$, the Picard-Lindelöf theorem ensures the existence and uniqueness of the solution to the IVP

$$\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \end{cases}$$





Initial Value Problem for Autonomous Dynamical Systems

The IVP for autonomous dynamical system comprises:

System of ODEs: $$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

ICs: $$y_1(a) = b_1 \qquad y_2(a) = b_2$$ at time $t = a$

We will consider solutions with $t \in (-\infty, \infty)$ including both future and past evolution

Under the hypothesis that a) the functions f_1, f_2
b) the partial derivatives $\frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial y_2}, \frac{\partial f_2}{\partial y_1}, \frac{\partial f_2}{\partial y_2}$

are continuous in $\mathbb{R}^2$

the Picard-Lindelöf theorem ensures the existence and uniqueness of the solution to the IVP.

Solution
$$\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \end{cases}$$