Feedback on Coursework #1

(1) Consider the following IVP

$$y' = \frac{(y+2)^{3/5}}{x+2} \qquad \& \quad y(2) = -2$$

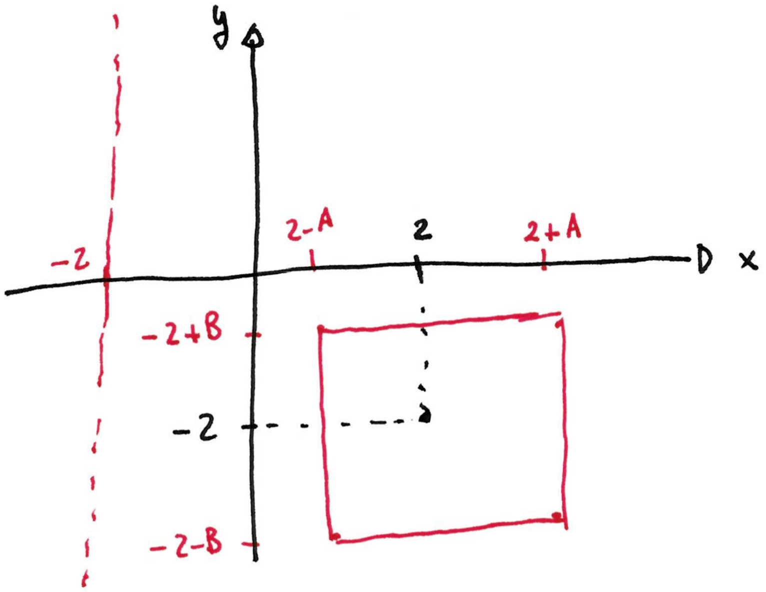
ODE: $\quad y' = f(x,y) = \dfrac{(y+2)^{3/5}}{x+2} \qquad \& \quad y(a) = b \; ; \; a = 2, \; b = -2$

(A) does the IVP satisfy the hypotheses of the Picard-Lindelöf theorem?

(B) how many (if any) solutions does the IVP have?

(A) IVP: $\qquad$ ODE $\quad y' = f(x,y) = \dfrac{(y+2)^{3/5}}{x+2}$

$\qquad$ IC $\quad y(2) = -2$



Consider the region

D

$|x-2| \leq A$

$|y+2| \leq B$

$A>0 \, , \, B>0$

$2-A \leq x \leq 2+A$

$-2-B \leq y \leq -2+B$

The hypotheses of the Picard-Lindelöf theorem are:

① $f(x,y)$ is continuous in D

$f(x,y) = \dfrac{(y+2)^{3/5}}{x+2}$ is continuous everywhere except for $x=-2$

In D we have $2-A \le x \le 2+A$

We must impose $\underbrace{-2 < 2-A}_{} \le x \Rightarrow 0 < A < 4$

$f(x,y)$ is continuous in D provided $0 < A < 4$

③ $A \le \dfrac{B}{M}$ where $M = \max_{(x,y)\in D} |f(x,y)|$

Since ① & ③ can be satisfied $\Rightarrow$ A solution exist in D

but there can be many solutions.

② Lipshitz condition $\dfrac{\partial}{\partial y} f(x,y)$ is bounded in D.

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} \frac{(y+2)^{3/5}}{(x+2)} = \frac{1}{(x+2)} \frac{3}{5} (y+2)^{-2/5} = \frac{3}{5} \frac{1}{(x+2)(y+2)^{2/5}}$$

diverges for $x \to -2$

$y \to -2$

There is no $A>0$, $B>0$ such that the Lipshitz condition can be satisfied.

(B) Let us solve IVP.

ODE: $$y' = \frac{(y+2)^{3/5}}{x+2} = g(y)\tilde{f}(x)$$ Separable!

$$g(y) = (y+2)^{3/5} \qquad \tilde{f}(x) = \frac{1}{x+2}$$

General solution

(i) $$\frac{dy}{dx} = \frac{(y+2)^{3/5}}{x+2} \longrightarrow \int \frac{dy}{(y+2)^{3/5}} = \int \frac{dx}{x+2} + C'$$

LHS: $$H(y) = \int \frac{dy}{(y+2)^{3/5}} = \frac{5}{2}(y+2)^{2/5}$$

RHS: $$F(x) = \int \frac{dx}{x+2} = \ln|x+2|$$

Implicit solution $\quad H(y) = F(x) + C'$

$$\frac{5}{2}(y+2)^{2/5} = \ln|x+2| + C'$$ where $C' \in \mathbb{R}$ is an arbitrary constant.

Explicit solution [check at home]

$$y = -2 \pm \left[\frac{2}{5}\left(\ln|x+2| + C\right)\right]^{5/2}$$

ii) Constant solutions $y = y^*$

with y^* is a root of $g(y)$.

In order to find y^* we need to impose

$$0 = g(y^*) = (y^*+2)^{3/5} \implies y^* = -2$$

Constant solution is $\boxed{y=-2}$

<u>Check to the solutions to the IVP</u>

i) $$y = -2 \pm \left[\frac{2}{5}\left(\ln|x+2| + C\right)\right]^{5/2}$$

Imposing $y(2) = -2$

$$-2 = -2 \pm \left[\frac{2}{5}\left(\ln|2+2| + C\right)\right]^{5/2} \implies C = -\ln 4$$

Two solutions to IVP

$$y = -2 \pm \left[\frac{2}{5}\left(\ln|x+2| - \ln 4\right)\right]^{5/2}$$

ii) $y = -2$ is a third solution to the IVP.