Variation of parameter method

Given a 2^{nd}-order linear inhomogeneous ODE of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

with two distinct roots $\lambda_1 \neq \lambda_2$ of the characteristic equation

$$H_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

The general solution $y_g(x)$ of the inhomogeneous ODE can be written as

$$y_g(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

and $y_p(x)$ is a particular solution of the inhomogeneous ODE.

With the variation of parameter method (see lecture week 5)

we have found a particular solution

$$y_p(x) = \frac{1}{a_2(\lambda_1 - \lambda_2)} \left\{ e^{\lambda_1 x} \underbrace{\int f(x)\, e^{-\lambda_1 x}\, dx}_{F_1(x)} - e^{\lambda_2 x} \underbrace{\int f(x)\, e^{-\lambda_2 x}\, dx}_{= F_2(x)} \right\}$$

where $y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ with c_1, c_2 arbitrary constants

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Solve the ODE

$$y'' - 3y' + 2y = e^{2x} \qquad f(x) = e^{2x}$$

Consider the characteristic equation of the corresponding homogeneous ODE $y'' - 3y' + 2y = 0$

$$M_2(\lambda) = \lambda^2 - 3\lambda + 2 = 0$$

→ Roots $\lambda = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2} = \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1 \end{cases}$

$\lambda_1 \neq \lambda_2$! We can apply the formula above.

The general solution to the inhomogeneous ODE will be of the form

$$y_g(x) = y_h(x) + y_p(x)$$

where $y_h(x) = c_1 e^{2x} + c_2 e^{x}$ where $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

ODE: $y'' - 3y' + 2y = e^{2x}$

$$y_p(x) = \frac{1}{1(2-1)} \left\{ e^{2x} F_1(x) - e^{x} F_2(x) \right\}$$

$$F_1(x) = \int f(x)\, e^{-\lambda_1 x} dx = \int e^{2x}\, e^{-2x}\, dx = \int dx = x$$

$$F_2(x) = \int f(x)\, e^{-\lambda_2 x}\, dx = \int e^{2x}\, e^{-x}\, dx = \int e^{x}\, dx = e^{x}$$

$$y_p(x) = x e^{2x} - e^{x} \cdot e^{x} = x e^{2x} - e^{2x} = (x-1)\, e^{2x}$$

$$y_g(x) = C_1 e^{2x} + C_2 e^{x} + (x-1)\, e^{2x}$$ with $C_1, C_2 \in \mathbb{R}$ arbitrary constant

Rearranging

$$y_g(x) = \underbrace{(C_1 - 1)}_{\tilde{C}_1}\, e^{2x} + \underbrace{C_2}_{= \tilde{C}_2}\, e^{x} + x e^{2x}$$

$$\Rightarrow y_g(x) = x e^{2x} + \tilde{C}_1 e^{2x} + \tilde{C}_2 e^{x}$$ with $\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}$ arbitrary constants

Note: The educated guess method cannot be used because $f(x) = e^{ax}$ with $a = 2 = \lambda_1$, i.e. a coincides with one of the roots of the characteristic equation.