Boundary Value Problem

Given A boundary value problem of 2^{nd}-order linear ODE comprised of

(A) Linear 2^{nd}-order ODE

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

with $a_2 \neq 0$, $a_2(x)$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) Linear boundary conditions

$$\alpha y'(x_1) + \beta y(x_1) = b_1$$

$$\gamma y'(x_2) + \delta y(x_2) = b_2$$

where $\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$

$(\alpha, \beta) \neq (0,0)$

$(\gamma, \delta) \neq (0,0)$

The Theorem of the Alternative

Consider the BVP for 2^{nd}-order linear ODE with linear BCs.

Only two alternatives are possible:

1 Either the BVP has a UNIQUE solution for

every choice of $f(x), b_1, b_2$

2 Or the corresponding homogeneous BVP has

INFINITE solutions

AND the inhomogeneous BVP have

(i) INFINITE SOLUTION for some choices of

$f(x), b_1, b_2$

(ii) NO SOLUTIONS for some choices of

$f(x), b_1, b_2$

Applications of the Theorem

- An inhomogeneous BVP has a UNIQUE SOLUTION

  if and only if

  the corresponding homogeneous BVP has a

  UNIQUE (trivial) solution

- If the corresponding homogeneous BVP has INFINITE solutions the inhomogeneous BVP can either have

  INFINITE SOLUTIONS

  NO SOLUTIONS

  (you need to check which option applies)

Example For which values of $b>0$ the following BVP has a UNIQUE solution?

$$y'' + b^2 y = \sin x \qquad \begin{cases} y(0) = 3 \\ y(1) = -2 \end{cases} \qquad \text{Note } b>0$$

This is an inhomogeneous BVP

For the theorem of the Alternative this BVP has a unique solution if and only if its corresponding homogeneous BVP has a unique solution.

Corresponding homogeneous BVP (*)

$$y'' + b^2 y = 0 \qquad \begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases} \qquad b>0$$

For which values of $b>0$ (*) has a unique solution?

(1) Solve $y'' + b^2 y = 0$ $\Rightarrow$ Characteristic equation

$$M_2(\lambda) = \lambda^2 + b^2 = 0$$

$\to$ Roots $\lambda^2 = -b^2$ $\begin{cases} \lambda_1 = ib \\ \lambda_2 = -ib \end{cases}$ Complex conjugate roots λ_1, λ_2

$$\lambda_1 = \alpha + i\beta \qquad \alpha = 0$$
$$\lambda_2 = \alpha - i\beta \qquad \beta = b$$

General solution

$$y_g(x) = A\cos bx + B\sin bx$$ where $A, B \in \mathbb{R}$ arbitrary constants.

② Impose BCs: $\begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases}$

$$0 = y(0) = A\overset{1}{\cos 0} + B\overset{0}{\sin 0} = A \implies \boxed{A = 0}$$

$$0 = y(1) = \underbrace{A\cos b}_{A=0} + B\sin b = B\sin b \implies \boxed{B\sin b = 0}$$

We have $A = 0$, $B\sin b = 0$

$B\sin b = 0$ implies either i) $\sin b = 0$ B is arbitrary

ii) $\sin b \neq 0 \implies B = 0$

In case (i) we have INFINITE solutions

in case (ii) we have ~~ONE~~ UNIQUE solution.

The corresponding homogeneous BVP (*) has a UNIQUE solution if and only if $\sin b \neq 0$

i.e. $b \neq \pi n$ where $n \in \mathbb{N}$

Using the Theorem of the Alternative we conclude that the inhomogeneous BVP

$$y'' + b^2 y = \sin x \qquad \begin{cases} y(0) = 3 \\ y(1) = -2 \end{cases}$$

has a UNIQUE solution for every $b>0$ such that

$$b \neq n\pi \qquad \text{with } n \in \mathbb{N}.$$

<u>Example</u> Find the smallest $b>0$ such that the BVP (**)

$$y'' + b^2 y = 0 \qquad \begin{cases} y(0) = 5 \\ y(1) = -5 \end{cases} \qquad b>0$$

has NO solutions

The corresponding homogeneous BVP is the same of the previous example

$$y'' + b^2 y = 0 \qquad \begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases} \qquad b>0$$

This BVP has INFINITE solutions if and only if $b = n\pi$ with $n \in \mathbb{N}$

For the Theorem of the Alternative if $b = n\pi$ with $n \in \mathbb{N}$ the inhomogeneous BVP (**) has either no solutions or infinite solutions.

We need to check directly when it has no solutions by going through the values

$$b = \pi, 2\pi, 3\pi, \ldots$$

(A) We check for $b = \pi$

The general solution to the ODE is

$y_g(x) = A\cos\pi x + B\sin\pi x$ with $A, B \in \mathbb{R}$ and arbitrary constants.

Imposing BCs: $\begin{cases} y(0) = 5 \\ y(1) = -5 \end{cases}$

$$5 = y(0) = A\cos 0 + B\sin 0 = A \quad \Rightarrow \boxed{A = 5}$$

$$-5 = y(1) = A\underbrace{\cos\pi}_{-1} + B\underbrace{\sin\pi}_{=0} = -A \quad \Rightarrow \boxed{A = 5}$$

B is arbitrary! The BVP has INFINITE SOLUTIONS.

$y_g(x) = 5\cos\pi x + B\sin\pi x$ where $B \in \mathbb{R}$ arbitrary constant.

(B) We check $b = 2\pi$

The general solution is $y_g(x) = A\cos 2\pi x + B\sin 2\pi x$

with $A, B \in \mathbb{R}$ arbitrary constants.

Imposing $\begin{cases} y(0) = 5 \\ y(1) = -5 \end{cases}$ we get

$5 = y(0) = A\cos 0 + B\sin 0 = A \Rightarrow \boxed{A = 5}$

$-5 = y(1) = A\cos 2\pi + B\sin 2\pi = A \Rightarrow \boxed{A = -5}$

The BVP has NO SOLUTIONS

The smallest value of $b > 0$ for which the BVP (**) has no solution is $\boxed{b = 2\pi}$