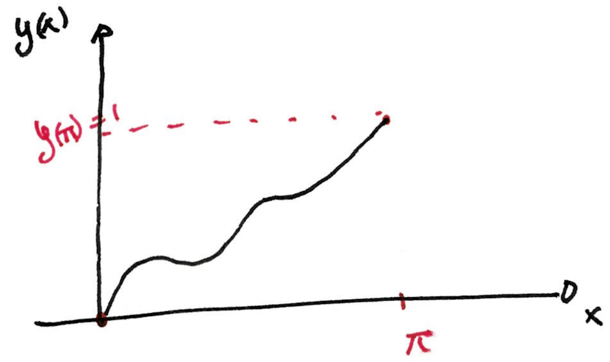
Motivation for the Theorem of the Alternative

Example ODE: $y'' + y = 0$ BCs: $\begin{cases} y(0) = 0 \\ y(\pi) = 1 \end{cases}$

Solving this BVP means finding $y(x)$ that solves the ODE and the BCs (IF ANY)



The corresponding homogeneous BVP is

ODE: $y'' + y = 0$ BCs: $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$

The BVP can admit only

1. One solution
2. No solution
3. Infinite solutions

The theorem of the Alternative will help us predict which of these cases apply to a given BVP

Let us first solve some BVP!

Example 1 Solve the BVP (inhomogeneous)

ODE: $y''+y=0$ BCs: $\begin{cases} y(0)=0 \\ y(\pi)=1 \end{cases}$

① We first find the general solution to the ODE $y''+y=0$.

Characteristic equation

$M_2(\lambda)=\lambda^2+1=0$

Roots

$\lambda^2=-1 \Rightarrow \begin{matrix} \lambda_1=i \\ \lambda_2=-i \end{matrix}$

Complex conjugate roots

$\begin{matrix} \lambda_1=\alpha+i\beta \\ \lambda_2=\alpha-i\beta \end{matrix}$ $\alpha=0$ $\beta=1$

General solution $y_g(x)=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

$\boxed{y_g(x)=A\cos x+B\sin x}$ with $A,B\in\mathbb{R}$ arbitrary constants.

② Impose the BCs $\begin{cases} y(0)=0 \\ y(\pi)=1 \end{cases}$

$$0=y(0)=A\underbrace{\cos 0}_{=1}+B\underbrace{\sin 0}_{=0}=A \Rightarrow \boxed{A=0}$$

$$1=y(\pi)=A\underbrace{\cos\pi}_{=-1}+B\underbrace{\sin\pi}_{=0}=-A \Rightarrow \boxed{A=-1}$$

There is NO SOLUTION to the BVP because the BCs cannot be simultaneously satisfied for any choice of A and B.

Example 2 Solve the BVP

ODE: $y'' + y = 0$ BCs: $\begin{cases} y(0) = 1 \\ y(\pi) = -1 \end{cases}$

(1) The general solution to the ODE $y'' + y = 0$ is

$$y_g(x) = A\cos x + B\sin x$$

with $A, B \in \mathbb{R}$ arbitrary constants (same as previous example)

(2) Impose the BCs $\begin{cases} y(0) = 1 \\ y(\pi) = -1 \end{cases}$

$$1 = y(0) = A\cos 0 + B\sin 0 = A \implies A = 1$$

$$-1 = y(\pi) = A\cos\pi + B\sin\pi = -A \implies A = 1$$

The BVP has INFINITE solutions

$$y(x) = \cos x + B\sin x$$

where $B \in \mathbb{R}$ is an arbitrary constant

Example 2 Solve the BVP (inhomogeneous)

ODE: $y''+y=0$

BCs: $\begin{cases} y(0)=1 \\ y(\pi)=-1 \end{cases}$

① The general solution to the ODE $y''+y=0$ is

$y_g(x) = A\cos x + B\sin x$ with $A, B \in \mathbb{R}$ arbitrary constants

(same as previous example)

② Impose the BCs: $\begin{cases} y(0)=1 \\ y(\pi)=-1 \end{cases}$

$$1 = y(0) = A\underbrace{\cos 0}_{=1} + B\underbrace{\sin 0}_{=0} = A \Rightarrow \boxed{A=1}$$

$$-1 = y(\pi) = A\underbrace{\cos\pi}_{=-1} + B\underbrace{\sin\pi}_{=0} = -A \Rightarrow \boxed{A=1}$$

The BVP has solution

$y(x) = \cos x + B\sin x$ where $B \in \mathbb{R}$ is an arbitrary constant.

The BVP has INFINITE solutions.

Example 3 Solve the homogeneous BVP corresponding to the BVP in examples 1 & 2

ODE: $y'' + y = 0$ BCs: $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$

(1) The general solution ~~to~~ to the ODE is

$y_g(x) = A\cos x + B\sin x$ with $A, B \in \mathbb{R}$ arbitrary constants

(2) We impose the BCs.

$$0 = y(0) = A\underbrace{\cos 0}_{1} + B\underbrace{\sin 0}_{0} = A \quad \Rightarrow \boxed{A = 0}$$

$$0 = y(\pi) = A\underbrace{\cos\pi}_{-1} + B\underbrace{\sin\pi}_{0} = -A \quad \Rightarrow \boxed{A = 0}$$

The solution to the BVP is

$y(x) = B\sin x$ with $B \in \mathbb{R}$ arbitrary constants

The BVP has INFINITE solution.

Finding Example 1 and example 2 are two examples of BVPs with no solution and infinite solutions respectively

The corresponding homogeneous BVP (which is the same) has INFINITE solutions

__Example 4__ ODE: $y''+y=0$ BCs: $\begin{cases} y(0)=1 \\ y(\frac{\pi}{2})=1 \end{cases}$

(inhomogeneous BVP)

① The general solution ~~is~~ to the ODE is

$$y_g(x) = A\cos x + B\sin x$$ where $A, B \in \mathbb{R}$ arbitrary constants

② We impose the BCs:

$$1 = y(0) = A\cos 0 + B\sin x = A \Rightarrow \boxed{A=1}$$

$$1 = y\left(\frac{\pi}{2}\right) = A\underbrace{\cos\frac{\pi}{2}}_{=0} + B\underbrace{\sin\frac{\pi}{2}}_{=1} = B \Rightarrow \boxed{B=1}$$

The BVP has a UNIQUE solution

$$\boxed{y(x) = \cos x + \sin x}$$

__Example 5__ Let us consider the homogeneous BVP corresponding to the BVP of example 4.

ODE: $y''+y=0$ BCs: $\begin{cases} y(0)=0 \\ y(\frac{\pi}{2})=0 \end{cases}$

① General solution to the ODE is

$$y_g(x) = A\cos x + B\sin x$$ with $A, B \in \mathbb{R}$ arbitrary constants.

(2) Imposing the BCs:

$$0 = y(0) = A\cos 0 + B\sin 0 = A \implies \boxed{A=0}$$

$$0 = y\left(\tfrac{\pi}{2}\right) = A\underbrace{\cos\tfrac{\pi}{2}}_{=0} + B\underbrace{\sin\tfrac{\pi}{2}}_{=1} = B \implies \boxed{B=0}$$

The BVP has a UNIQUE (trivial) solution

$$\boxed{y=0}$$

Finding: Example 4 is an example of BVP with a UNIQUE solution whose corresponding homogeneous BVP (Examples 5) has a UNIQUE (trivial solution)

How general are these findings?

(see discussion of the Theorem of the Alternative on lesson 3 week 6)