The IVP of a 2^{nd}-order linear ODE

The IVP of a 2^{nd}-order linear ODE comprises of

(A) ODE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

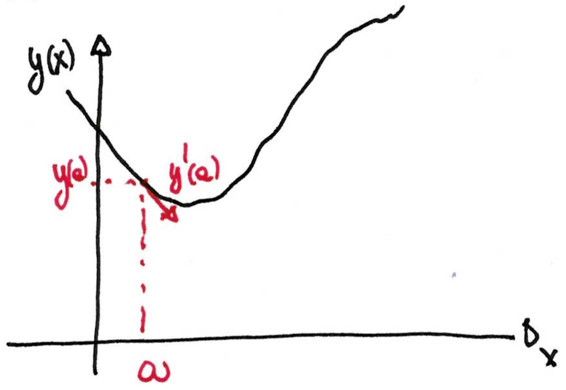
with $a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [A, B]$

(B) I.C.s

$$y(a) = b_1$$

$$y'(a) = b_2 \qquad \text{with } a \in [A, B]$$

The I.C.s determine the value of the function $y(x)$ and its derivative AT THE SAME POINT!



Boundary Value Problem (BVP)

The boundary value problem (BVP) consists of

(A) ODE:

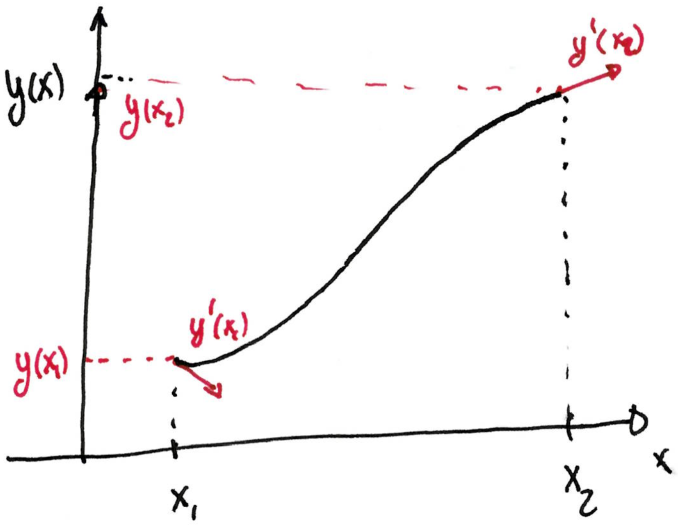
$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x)$$

with $a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) Two boundary conditions specified at the two endpoints

$x = x_1$ and $x = x_2$ with $x_1 < x_2$

We will consider linear boundary conditions



Linear boundary conditions

$$\alpha\, y'(x_1) + \beta\, y(x_1) = b_1$$

$$\gamma\, y'(x_2) + \delta\, y(x_2) = b_2$$

where $\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$

$(\alpha, \beta) \neq (0,0)$

$(\gamma, \delta) \neq (0,0)$

Examples

$$\begin{cases} y(x_1) = b_1 \\ y(x_2) = b_2 \end{cases} \qquad \alpha = 0 \quad \beta = 1 \quad b_1 = b_1 \qquad \gamma = 0 \quad \delta = 1 \quad b_2 = b_2 \quad ✓$$

$$\begin{cases} y'(x_1) = b_1 \\ y'(x_2) = b_2 \end{cases} \qquad \alpha = 1 \quad \beta = 0 \qquad \gamma = 1 \quad \delta = 0 \quad ✓$$

$$\begin{cases} 2y'(x_1) + 3y(x_1) = 5 \\ y'(x_2) + 4y(x_2) = 3 \end{cases} \qquad \alpha = 2 \quad \beta = 3 \quad b_1 = 5 \qquad \gamma = 1 \quad \delta \cancel{\beta} = 4 \quad b_2 = 3 \quad ✓$$

$$\begin{cases} y'(x_1) + y(x_2) = 5 \\ y'(x_2) = 5 \end{cases}$$

Are these BCs? No!

Solving BVP

Solving a Boundary Value Problem implies finding the functions (if any) $y(x)$ that satisfy

(A) ODE

(B) BCs

in the interval $x \in [x_1, x_2]$

> The BVP can have
> - no solution
> - one solution
> - infinite solutions

Homogeneous and Inhomogeneous BCs.

A set of linear boundary conditions

$$\alpha y'(x_1) + \beta y(x_1) = b_1$$
$$\gamma y'(x_2) + \delta y(x_2) = b_2$$

is called HOMOGENEOUS if and only if $\begin{cases} b_1 = 0 \\ b_2 = 0 \end{cases}$

A set of BCs ~~is~~ that is not homogeneous is called INHOMOGENEOUS

<u>Example</u> a) $\begin{cases} y(1) + 3y'(1) = 0 \\ y(3) = 3 \end{cases}$ $\quad b_1 = 0$, $b_2 = 3$

<u>Inhomogeneous BCs</u>

The corresponding homogeneous BC of an inhomogeneous BC is obtained by putting $b_1 = 0$, $b_2 = 0$.

<u>Example</u> b) The homogeneous BC corresponding to the BC in example a is

$$\begin{cases} y(1) + 3y'(1) = 0 \\ y(3) = 0 \end{cases}$$

Homogeneous and Inhomogeneous BVP

Let us consider the BVP

(A) ODE

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x)$$

with $a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) Linear BCs

$$\begin{cases} \alpha y'(x_1) + \beta y(x_1) = b_1 \\ \gamma y'(x_2) + \delta y(x_2) = b_2 \end{cases}$$

where $\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$
$(\alpha, \beta) \neq (0,0)$
$(\gamma, \delta) \neq (0,0)$

The BVP is HOMOGENEOUS if and only if

(A) the ODE is homogeneous i.e. $f(x) = 0$

(B) the BCs are homogeneous i.e. $b_1 = b_2 = 0$

The BVP that ~~is~~ are not homogeneous are called INHOMOGENEOUS

Given an inhomogeneous BVP the corresponding homogeneous BVP can be obtained by putting

$$f(x) = 0 \quad \text{and} \quad b_1 = b_2 = 0$$

Example

ODE: $y'' + y = e^x$ $\qquad f(x) = e^x$

BCs: $\begin{cases} y(0) = 1 \\ y'(2) = 3 \end{cases}$ $\qquad b_1 = 1$, $b_2 = 3$

The BVP is inhomogeneous.

c) ODE: $e^x y'' + \sin x \, y' - \tan 2x = 0$ $\qquad f(x) = \tan 2x$

BCs: $\begin{cases} y(1) = 0 \\ y(2) = 3 \end{cases}$ $\qquad b_1 = 0$, $b_2 = 3$

The BVP is inhomogeneous

d) ODE: $e^x y'' + \sin x \, y' = 0$ $\qquad f(x) = 0$

BCs: $\begin{cases} y(1) = 0 \\ y(2) = 3 \end{cases}$ $\qquad b_1 = 0$, $b_2 = 3$

The BVP is inhomogeneous.

The corresponding homogeneous BVP to c) and d) is

ODE: $e^x y'' + \sin x \, y' = 0$

BCs: $\begin{cases} y(1) = 0 \\ y(2) = 0 \end{cases}$