Educated guess method

The variation of parameter method can be applied to any ODE of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (1)$$

The educated guess method can only be applied to the inhomogeneous ODE of the type (1) which have

$$f(x) = p(x)\, e^{ax}$$

- where $p(x)$ is a polynomial of degree k
- $a \neq \lambda_1, \lambda_2$ where λ_1, λ_2 are the roots of the characteristic equation

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

In these conditions the educated guess method provides one particular solution to (1) that we will indicate with $y_p(x)$

The general solution to (1) will be

$$y(x) = y_h(x) + y_p(x)$$

Particular solution $y_p(x)$

The particular solution $y_p(x)$ can be written as

$$y_p(x) = Q(x) \cdot e^{ax} \qquad (3)$$

where $Q(x)$ is a polynomial of degree k (the same degree of $p(x)$)

$$Q(x) = d_k x^k + d_{k-1} x^{k-1} + \dots \qquad + d_1 x + d_0$$

where the coefficients $d_k, d_{k-1}, d_{k-2} \dots d_1, d_0$ can be determined by imposing that (3) is a solution to (1)

We will use an example to illustrate this method

<u>Example</u> $\quad y'' + 2y' - 3y = x e^{2x}$

This equation is of the type $a_2 y'' + a_1 y' + a_0 y = f(x)$

$$a_2 = 1, \quad a_1 = 2, \quad a_0 = -3, \quad f(x) = x e^{2x}$$

1. Check if we can use the educated guess method.

$$f(x) = p(x) e^{ax} \qquad p(x) = x \qquad a = 2$$

(A) $p(x)$ is a polynomial of degree $k = 1$ ✓

(B) $a = 2 \neq \lambda_1, \lambda_2$ where λ_1, λ_2 are the roots of

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

The characteristic equation reads

$$\lambda^2+2\lambda-3=0$$

$$\lambda=\frac{-2\pm\sqrt{4+12}}{2}=\frac{-2\pm\sqrt{16}}{2}=\frac{-2\pm4}{2}=\begin{cases}1=\lambda_2\\-3=\lambda_1\end{cases}$$

$a=2\neq\lambda_1,\lambda_2$ ✓

② Find the general solution to the homogeneous problem

The roots of $M_2(\lambda)=0$ are two real distict $\lambda_1=-3\neq\lambda_2=1$

(A) $$y_h(x)=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$$

$$y_h(x)=c_1e^{-3x}+c_2e^{x}$$, $c_1,c_2\in\mathbb{R}$ and arbitrary constants.

③ Use the educated guess method to find the particular solution to

$$y''+2y'-3y=xe^{2x}$$

We look for a solution $y_p(x)$ of the type

$$y_p(x)=Q(x)e^{ax}=(d_1x+d_0)e^{2x}$$

To this end we need to calculate $y_p'(x)$ and $y_p''(x)$

$$y_p'(x)=\frac{d}{dx}\left[(d_1x+d_0)e^{2x}\right]=2(d_1x+d_0)e^{2x}+d_1e^{2x}$$

$$\boxed{y_p'(x)=(2d_1x+d_1+2d_0)e^{2x}}$$ (*)

$$y_p''(x) = \frac{d}{dx}\, y_p'(x) = \frac{d}{dx}\left[(2d_1 x + d_1 + 2d_0)\, e^{2x}\right] =$$

$$y_p''(x) = 2\,(2d_1 x + d_1 + 2d_0)\, e^{2x} + 2 d_1 e^{2x}$$

$$\boxed{y_p''(x) = (4 d_1 x + 4 d_1 + 4 d_0)\, e^{2x}} \quad (**)$$

Inserting (*) and (**) into (1) we get

$$y'' + 2y' - 3y = x e^{2x} \qquad (1)$$

$$(4 d_1 x + 4 d_1 + 4 d_0)\, \cancel{e^{2x}} + 2\,(2 d_1 x + d_1 + 2 d_0)\, \cancel{e^{2x}} - 3\,(d_1 x + d_0)\, e^{2x} = x\, \cancel{e^{2x}}$$

Rearranging

$$5 d_1 x + 6 d_1 + 5 d_0 = x$$

Matching the coefficients in the polynomials

$$\begin{cases} 5 d_1 = 1 \\ 6 d_1 + 5 d_0 = 0 \end{cases} \Rightarrow \begin{cases} d_1 = 1/5 \\ d_0 = -\frac{6 d_1}{5} = -\frac{6}{25} \end{cases}$$

The particular solution to (1) is given by

$$y_p(x) = (d_1 x + d_0)\, e^{2x}$$

$$\boxed{y_p(x) = \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x}}$$

⑤ The general solution $y_g(x)$ of the inhomogeneous ODE is given by

$$y_g(x) = y_h(x) + y_p(x)$$

$$\boxed{y_g(x) = c_1 e^{-3x} + c_2 e^{x} + \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x}}$$

with $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

<u>Note</u> The educated guess method can also be applied to inhomogeneous ODE of the type.

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

with

$$f(x) = p(x) \cos(ax)$$

or

$$f(x) = p(x) \sin(ax)$$

where $\cos(ax)$ and $\sin(ax)$ are not solutions to the homogeneous problem

In this case the particular solution $y_p(x)$ will have the form

$$y_p(x) = Q(x)\left[A \cos(ax) + B \sin(ax)\right]$$

where $Q(x)$ is a polynomial of the same degree of $p(x)$

$$Q(x) = d_k x^k + d_{k-1} x^{k-1} + \dots \qquad d_1 x + d_0$$

In this case the application of the method follows the same steps of the example above.