Variation of parameter method

Step I Finds the general solution to the homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

We assume that the characteristic equation has two distinct solutions $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \quad , \quad M_2(\lambda_1) = M_2(\lambda_2) = 0$$

The general solution is

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

where $c_1, c_2 \in \mathbb{R}$ and arbitrary constants.

Step II Consider the inhomogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (1)$$

We look for solutions of the type

$$y(x) = c_1(x) e^{\lambda_1 x} + c_2(x) e^{\lambda_2 x}$$

and we determine under which conditions this is a solution to (1)

Derivation - Not examinable

Note: $$\frac{d}{dx}\left(c_i(x)e^{\lambda_i x}\right) = \left(\frac{d}{dx}c_i(x)\right)e^{\lambda_i x} + c_i(x)\underbrace{\left(\frac{d}{dx}e^{\lambda_i x}\right)}_{=\lambda_i e^{\lambda_i x}}$$

$$\boxed{\frac{d}{dx}\left(c_i(x)e^{\lambda_i x}\right) = c_i'(x)e^{\lambda_i x} + \lambda_i c_i(x) e^{\lambda_i x}} \qquad (3)$$

Let us calculate $y'(x)$ and $y''(x)$

$$y'(x) = \frac{d}{dx}\underbrace{\left[c_1(x)e^{\lambda_1 x} + c_2(x)e^{\lambda_2 x}\right]}_{=y(x)}$$

Using (3) we get

$$y'(x) = c_1'(x)e^{\lambda_1 x} + \lambda_1 c_1(x)e^{\lambda_1 x} + c_2'(x)e^{\lambda_2 x} + \lambda_2 c_2(x)e^{\lambda_2 x}$$

Let us impose

$$\boxed{c_1'(x)e^{\lambda_1 x} + c_2'(x)e^{\lambda_2 x} = 0 \qquad (4)}$$

Therefore we get

$$\boxed{y'(x) = c_1(x)\lambda_1 e^{\lambda_1 x} + c_2(x)\lambda_2 e^{\lambda_2 x} \qquad (5)}$$

Differentiating Eq. (5) we get

$$y''(x) = \frac{d}{dx} y'(x) \underset{\substack{\uparrow \\ \text{Eq. (5)}}}{=} \frac{d}{dx}\left[c_1(x)\lambda_1 e^{\lambda_1 x} + c_2(x)\lambda_2 e^{\lambda_2 x} \right] =$$

$$y'' = \lambda_1 \frac{d}{dx}\left(c_1(x) e^{\lambda_1 x} \right) + \lambda_2 \frac{d}{dx}\left(c_2(x) e^{\lambda_2 x} \right)$$

Recalling Eq. (3) $\quad \frac{d}{dx}\left(c_i(x) e^{\lambda_i x} \right) = c_i'(x) e^{\lambda_i x} + \lambda_i c_i(x) e^{\lambda_i x}$

we get

$$\boxed{y'' = \lambda_1 \left[c_1'(x) e^{\lambda_1 x} + \lambda_1 c_1(x) e^{\lambda_1 x} \right] + \lambda_2 \left[c_2'(x) e^{\lambda_2 x} + \lambda_2 c_2(x) e^{\lambda_2 x} \right]} \quad (6)$$

We insert the expression of $y(x)$, $y'(x)$ and $y''(x)$ as a function of $c_1(x)$ and $c_2(x)$ into our inhomogeneous linear ODE.

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

getting

$$a_2 \underbrace{\left(\lambda_1 c_1'(x) e^{\lambda_1 x} + \lambda_1^2 c_1(x) e^{\lambda_1 x} + \lambda_2 c_2'(x) e^{\lambda_2 x} + \lambda_2^2 c_2(x) e^{\lambda_2 x} \right)}_{= y''} +$$

$$a_1 \underbrace{\left(c_1(x) \lambda_1 e^{\lambda_1 x} + c_2(x) \lambda_2 e^{\lambda_2 x} \right)}_{= y'} + a_0 \underbrace{\left(c_1(x) e^{\lambda_1 x} + c_2(x) e^{\lambda_2 x} \right)}_{= y} = f(x)$$

Reordering we get

$$c_1(x)\, e^{\lambda_1 x} \underbrace{\overbrace{\left(a_2\lambda_1^2 + a_1\lambda_1 + a_0\right)}^{M_2(\lambda_1)=0}}_{=0} + c_2(x)\, e^{\lambda_2 x} \underbrace{\overbrace{\left(a_2\lambda_2^2 + a_1\lambda_2 + a_0\right)}^{M_2(\lambda_2)=0}}_{=0} +$$

$$+ \left(a_2 \lambda_1 c_1'(x)\, e^{\lambda_1 x} + a_2 \lambda_2 c_2'(x)\, e^{\lambda_2 x} \right) = f(x)$$

$$\Rightarrow \quad a_2\left(\lambda_1 c_1'(x)\, e^{\lambda_1 x} + \lambda_2 c_2'(x)\, e^{\lambda_2 x}\right) = f(x)$$

$$\lambda_1 c_1'(x)\, e^{\lambda_1 x} + \lambda_2 c_2'(x)\, e^{\lambda_2 x} = \frac{f(x)}{a_2} \qquad \text{Eq } (7)$$

which is obtained under the assumption

$$c_1'(x)\, e^{\lambda_1 x} + c_2'(x)\, e^{\lambda_2 x} = 0 \qquad (8)$$

Solving (7-8) for $c_1'(x)$, $c_2'(x)$ we obtain

$$\begin{cases} c_1'(x) = \dfrac{-1}{a_2(\lambda_2 - \lambda_1)}\, f(x)\, e^{-\lambda_1 x} \\[2em] c_2'(x) = \dfrac{1}{a_2(\lambda_2 - \lambda_1)}\, f(x)\, e^{-\lambda_2 x} \end{cases}$$

Integrating with respect to x

$$c_1(x) = \frac{1}{a_2(\lambda_1-\lambda_2)}\left[\int f(x)\, e^{-\lambda_1 x} + C_1\right]$$

$$c_2(x) = \frac{-1}{a_2(\lambda_1-\lambda_2)}\left[\int f(x)\, e^{-\lambda_2 x} + C_2\right]$$

Where $C_1, C_2 \in \mathbb{R}$ are arbitrary constant.

Inserting these expression into the solution

$$y(x) = c_1(x)\, e^{\lambda_1 x} + c_2(x)\, e^{\lambda_2 x}$$

we get the general solution

$$y(x) = y_h(x) + y_p(x)$$

where $\quad y_h(x) = d_1 e^{\lambda_1 x} + d_2 e^{\lambda_2 x} \quad$ with

$$d_1 = \frac{C_1}{a_2(\lambda_1-\lambda_2)} \qquad d_2 = \frac{-C_2}{a_2(\lambda_1-\lambda_2)}$$ they are arbitrary constants.

and the particular solution $y_p(x)$ is given by

$$y_p(x) = \frac{1}{a_2(\lambda_1 - \lambda_2)} \left[e^{\lambda_1 x} \int f(x)\, e^{-\lambda_1 x}\, dx - e^{\lambda_2 x} \int f(x)\, e^{-\lambda_2 x}\, dx \right]$$

$\square$