Euler type ODEs

These are 2^{nd}-order homogeneous ODEs of the type

$$\boxed{a x^2 y'' + b x y' + c y = 0} \qquad (1)$$

with $a, b, c \in \mathbb{R}$

These are NOT ODEs with constant coefficients but they are <u>reducible</u> to ODEs with constant coefficient

This can be achieved by setting

$x = e^t$ or $t = \ln x$ with $x > 0$

and considering the function

$$z = z(t) = y(x(t))$$

<u>Proposition</u> The function $z = z(t)$ defined as $z = y(x(t))$ with $x = e^t$ obeys the 2^{nd}-order linear homogeneous ODE with constant coefficients.

$$\boxed{a \ddot{z} + (b - a) \dot{z} + c z = 0} \qquad (2)$$

Proof   Let us calculate $\dot{z}$ and $\ddot{z}$

We have

$$\dot{z} = \frac{d\,z(t)}{dt} = \frac{d}{dt}\,y(x(t)) \overset{\text{chain rule}}{=} \frac{d\,y(x)}{dx}\,\frac{dx}{dt} \overset{x=e^t}{=} y'\,\frac{de^t}{dt} = y'\,e^t$$

$$\boxed{\dot{z} = y' e^t}$$

$$\ddot{z} = \frac{d\,\dot{z}(t)}{dt} = \frac{d}{dt}\left(y'e^t\right) \overset{\text{product rule}}{=} \left(\frac{dy'}{dt}\right)e^t + y'\,\frac{de^t}{dt} = \left(\frac{dy'}{dt}\right)e^t + y'e^t$$

$$\ddot{z} = \left(\frac{dy'}{dt}\right)e^t + y'\,e^t \qquad (*)$$

Now using the chain rule

$$\frac{dy'}{dt} = \frac{d}{dt}\,y'(x(t)) \overset{\text{chain rule}}{=} \frac{d\,y'(x)}{dx}\cdot\frac{dx}{dt} = y''\,\frac{de^t}{dt} = y''\,e^t \qquad (**)$$

Therefore introducing $(**)$ into $(*)$ we get

$$\ddot{z} = \underbrace{\left(y''e^t\right)}_{=\frac{dy'}{dt}}e^t + y'e^t = y''\,e^{2t} + \underbrace{y'\,e^t}_{=\dot{z}}$$

$$\ddot{z} = y''e^{2t} + \dot{z} \implies \boxed{\ddot{z} - \dot{z} = y''e^{2t}}$$

Therefore we have

$$\begin{cases} \dot{z} = y' e^t \\ \ddot{z} - \dot{z} = y'' e^{2t} \end{cases} \qquad x = e^t \quad \Rightarrow \quad \begin{cases} \dot{z} = y' x \\ \ddot{z} - \dot{z} = y'' x^2 \end{cases} \qquad (***)$$

Recall the Euler equation

$$a \underbrace{x^2 y''}_{\ddot{z}-\dot{z}} + b \underbrace{x y'}_{\dot{z}} + c y = 0 \qquad y(x(t)) = z$$

Inserting (***) into the Euler equation we get

$$a(\ddot{z} - \dot{z}) + b\dot{z} + cz = 0$$

$$\boxed{a\ddot{z} + (b-a)\dot{z} + cz = 0}$$

□

Solving Euler ODEs

Example $\quad x^2 y'' - 4xy' + 6y = 0$

① Recognize it is an Euler ODE of the type

$$a x^2 y'' + b x y' + c y = 0$$

$a = 1 \quad , \quad b = -4 \quad , \quad c = 6 \quad ✓$

(2) Define $z = z(t)$ given by $z = y(x(t))$ where $x > 0$

$$x = e^t \quad \Rightarrow \quad t = \ln x$$

$z(t)$ satisfies
$$a\ddot{z} + (b-a)\dot{z} + cz = 0$$
$$\ddot{z} + (-4-1)\dot{z} + 6z = 0$$
$$\ddot{z} - 5\dot{z} + 6z = 0$$

(3) Solve $\ddot{z} - 5\dot{z} + 6z = 0$

Characteristic equation

$$M_2(\lambda):\ \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2} \begin{cases} \frac{5+1}{2} = 3 = \lambda_1 \\ \frac{5-1}{2} = 2 = \lambda_2 \end{cases}$$

We have two distinct roots $\lambda_1 = 3 \neq \lambda_2 = 2$

(A) The general solution

$$z_g(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$z_g(t) = c_1 e^{3t} + c_2 e^{2t}$$ where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

(4) To find $y(x)$ we impose

$$y_g(x) = z_g(t)\Big|_{t = \ln x}$$ with $x > 0$

In our case

$$z_g(t) = c_1 e^{3t} + c_2 e^{2t}$$

putting $t = \ln x$

$$y_g(x) = z_g(t)\Big|_{t=\ln x} = c_1 e^{3\ln x} + c_2 e^{2\ln x} = c_1 x^3 + c_2 x^2$$

$$y_g(x) = c_1 x^3 + c_2 x^2$$

with $x > 0$

$c_1, c_2 \in \mathbb{R}$ and arbitrary constants.

Variation of parameter method (2nd-order ODEs)

Examinable
- Method -

The variation of parameter method solves the 2nd-order linear inhomogeneous ODE with constant coefficients of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (1)$$

where $f(x) \neq 0$, and $a_2, a_1, a_0 \in \mathbb{R}$ with $a_2 \neq 0$

The method provides the general solution to (1) which is of the form

$$\boxed{y_g(x) = y_h(x) + y_p(x)}$$

where $y_h(x)$ is the general solution to the homogeneous problem corresponding to (1)

$y_p(x)$ is a particular solution of (1)

- We assume that the characteristic equation

$$H_2(\lambda): \; a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

has two distinct roots $\lambda_1 \neq \lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$.

The general solution to the homogeneous problem $y_h(x)$ reads

$$y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$ with $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

The variation of parameter method provides a particular solution $y_p(x)$ to Eq. (1) given by

$$\boxed{y_p(x) = \frac{1}{a_2(\lambda_1-\lambda_2)} \left[e^{\lambda_1 x} \int f(x)\, e^{-\lambda_1 x}\, dx - e^{\lambda_2 x} \int f(x)\, e^{-\lambda_2 x}\, dx \right]}$$

Note: the usefulness of this method relies on the ability to calculate the two integrals

$$\int f(x)\, e^{-\lambda_i x}\, dx$$ for $i = 1$ and $i = 2$.