• $y' = x^2 + y^2$ & $y(0) = 1$

I.C. $y(a) = b$

ODE: $y' = f(x,y) = x^2 + y^2$

$a = 0$, $b = 1$

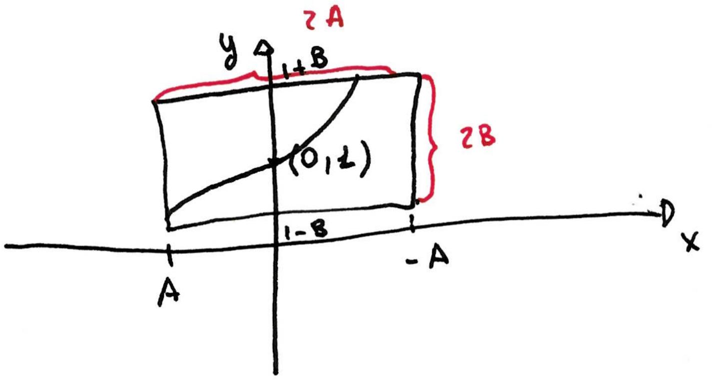
$(0,1)$

Consider D: $|x - a| \leq A$, $|y - b| \leq B$ where $A > 0$, $B > 0$

$|x| \leq A$ $|y - 1| \leq B$

$-A \leq x \leq A$ $-B \leq y - 1 \leq B$

$1 - B \leq y \leq 1 + B$



Under which condition on A and B the Picard-Lindelöf theorem guarantees that the solution to the IVP exists and is unique?

$y' = f(x,y)$ & $y(a) = b$

① $f(x,y)$ is continuous in D

② $\frac{\partial}{\partial y}(f(x,y))$ is bounded in D - Note that if $\frac{\partial}{\partial y} f$ is continuous in D then it is certainly bounded in D

③ $A \leq \frac{B}{M}$ where $M = \max_{(x,y) \in D} |f(x,y)|$

① $f(x,y) = x^2 + y^2$ is continuous in D ✓

② $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{\partial}{\partial y}(x^2+y^2) = 2y$ is continuous in D ✓

③ $A \leq \dfrac{B}{M}$ where $M = \max\limits_{(x,y)\in D} |f(x,y)| = \max\limits_{(x,y)\in D} (x^2+y^2) =$

$$M = \max_{(x,y)\in D} x^2 + \max_{(x,y)\in D} y^2 = \max_{-A\leq x\leq A} x^2 + \max_{1-B\leq y\leq 1+B} y^2$$

$$M = A^2 + (1+B)^2$$

$$\boxed{A \leq \frac{B}{A^2+(1+B)^2}}$$

This condition is the one that guarantees existence and uniqueness of the solution to I.V.P.

$$A\left(A^2 + (1+B)^2\right) \leq B$$

$$A^3 + A\left(B^2 + 2B + 1\right) \leq B$$

$$AB^2 + (2A-1)B + A^3 + A \leq 0$$

$$B_1, B_2$$

$$B_1(A) \leq B \leq B_2(A)$$

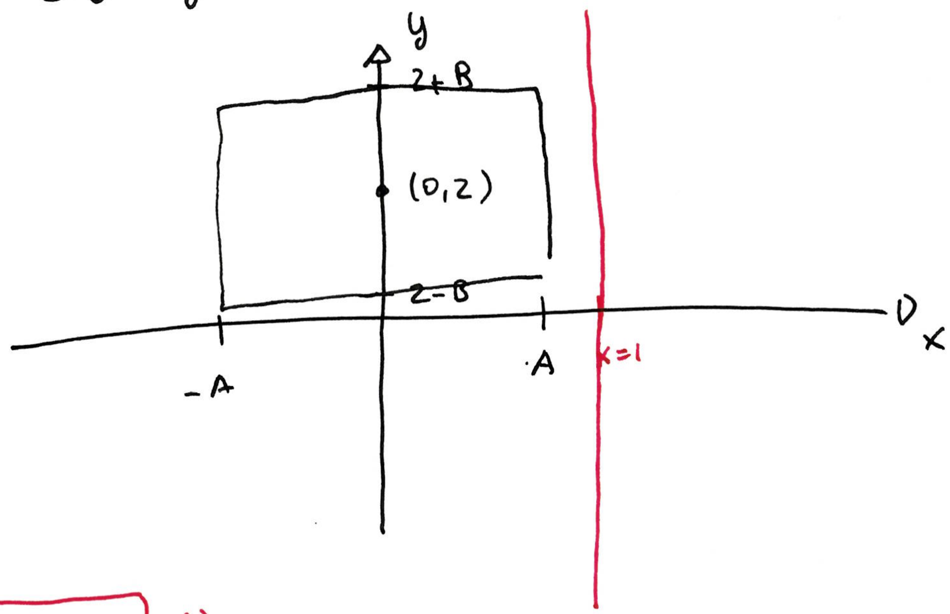
$$y' = \frac{x}{x-1}\,y \qquad \& \qquad y(0)=2$$

I.C. $y(a)=b$

$a=0$, $b=2$

D: $|x| \leq A$ $\quad |y-2| \leq B$ $\quad A>0,\ B>0$

$-A \leq x \leq A$ $\qquad 2-B \leq y \leq 2+B$



① $f(x,y) = \dfrac{xy}{x-1}$

$-A \leq x \leq A < 1$

$$\boxed{0 < A < 1} \quad (1)$$

Provided (1) holds then $f(x,y)$ is continuous in D

② $\dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial}{\partial y} \dfrac{xy}{x-1} = \dfrac{x}{x-1}$ is continuous in D provided (1) holds.

③ $A \leq \dfrac{B}{M}$ where $M = \max\limits_{(x,y)\in D} |f(x,y)| = \max\limits_{(x,y)\in D} \left|\dfrac{x}{x-1}\,y\right|$

$$= \underbrace{\max_{-A \leq x \leq A} \left|\frac{x}{x-1}\right|}_{\frac{A}{1-A}} \cdot \underbrace{\max_{2-B \leq y \leq 2+B} |y|}_{2+B}$$

Provided (1) and (2) holds the P. L. theorem ensures existence and uniqueness of the solution to the I.V.P.

$$M = \frac{A}{1-A}(2+B) \Rightarrow A \leq \frac{B\,(1-A)}{A\,(2+B)} \Rightarrow \boxed{\frac{A^2}{1-A} \leq \frac{B}{2+B}} \quad (2)$$

(1) $f(x,y) = \dfrac{xy}{x-1}$ continuous provided $x \neq 1$

$$-A \leq x \leq A < 1$$

$$\boxed{0 < A < 1} \qquad (1)$$

(2) $\dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{xy}{x-1}\right) = \dfrac{x}{x-1}$ is continuous provided $x \neq 1$

$$0 < A < 1$$

(3) $A \leq \dfrac{B}{M}$ where $M = \max\limits_{(x,y)\in D} |f(x,y)| = \max\limits_{(x,y)\in D} \left|\dfrac{xy}{x-1}\right| =$

$$= \max_{-A \leq x \leq A} \left|\frac{x}{x-1}\right| \cdot \max_{2-B \leq y \leq 2+B} |y|$$

$$= \frac{A}{1-A}(2+B)$$

$$A \leq \frac{B}{A(2+B)}(1-A) \Longrightarrow \boxed{\frac{A^2}{1-A} \leq \frac{B}{2+B}} \qquad (2)$$

Provided A, B satisfy (1) and (2) the hypotheses of the P. L. theorem are satisfied, therefore the IVP has a unique solution.